Instructions

- Of the following six problems, please do five of your choice. Please do not include solutions to all six problems; if you do so, then only the first five will be graded. It is recommended that you mark an X through the problem you do not want graded.
- Each problem is worth 20 points.
- Please start each problem on a new page labeled with the problem number.
- Write your secret code on each page and number all pages in order.
- Justify your answers and show all your work.
- In what follows, $\mathbb{R}$ and $\mathbb{C}$ denote the set of real and complex numbers, respectively.

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1. Let $\mathbf{x} \in \mathbb{R}^n$ be a vector and $A, B \in \mathbb{R}^{n\times n}$ be two square matrices. Let further $\|\cdot\|_n$ denote a vector norm on $\mathbb{R}^n$.

(a) (5 points) Define the induced matrix norm (or operator norm) $\|\cdot\|_{n,n}$ on $\mathbb{R}^{n\times n}$ corresponding to the vector norm $\|\cdot\|_n$.

(b) (2 point) Show that $\|A\mathbf{x}\|_n \le \|A\|_{n,n}\,\|\mathbf{x}\|_n$.

(c) (5 points) Prove that $\|AB\|_{n,n} \le \|A\|_{n,n}\|B\|_{n,n}$, where $A \in \mathbb{R}^{m\times n}$ and $B \in \mathbb{R}^{n\times r}$.

(d) (3 points) Prove that $\|I\|_{n,n} = 1$, where I is the $n \times n$ identity matrix.

(e) (5 points) Prove that the condition number A with respect to the induced matrix norm $\|\cdot\|_{n,n}$ satisfies $\kappa(A) = \|A\|_{n,n}\|A^{-1}\|_{n,n} \ge 1$.

2. Consider the linear system $A\mathbf{x} = \mathbf{b}$, where $A \in \mathbb{R}^{n\times n}$ is invertible.

(a) (8 points) Using appropriate splittings of matrix A of the form $A = B + (A - B)$, i.e. by selecting an appropriate B matrix, derive the following formulation of the Jacobi iterative algorithm,

$$\mathbf{x}^{(k+1)} = R\mathbf{x}^k + \mathbf{c}, \qquad k = 0, 1, 2, \dots,$$

by finding R and $\mathbf{c}$ in terms of A, B, and $\mathbf{b}$.

(b) (8 points) Repeat part (a) and derive the formulation of Gauss-Seidel iterative algorithm. Note that the matrix B and hence R and $\mathbf{c}$ for Gauss-Seidel are different from Jacobi.

(c) (4 points) State (without proving) a condition on the matrix A that guarantees the convergence of both Jacobi and Gauss-Seidel methods.

3. Let the singular value decomposition of an approximately low-rank matrix $A \in \mathbb{R}^{n\times n}$ be given by

$$A = U\Sigma V^\top, \qquad \Sigma = \begin{pmatrix} \sigma_1 & & & & & \\ & \ddots & & & & \\ & & \sigma_r & & & \\ & & & \sigma_{r+1} & & \\ & & & & \ddots & \\ & & & & & \sigma_n \end{pmatrix},$$

where

$$\sigma_1 > \sigma_2 > \ldots > \sigma_r > \quad \varepsilon \quad \geq \sigma_{r+1} \geq \ldots \geq \sigma_n \geq 0, \qquad 0 \leq \varepsilon \ll 1.$$

Suppose that $\hat{A} \in \mathbb{R}^{n\times n}$ is the r-rank approximation of A, obtained by replacing the singular values $\sigma_{r+1}, \ldots, \sigma_n$ that are smaller than ε by zero.

(a) (10 points) Explain how the cost of matrix-vector multiplication $A\mathbf{x}$ can be reduced by the approximation $\hat{A}\mathbf{x}$. You will need to count the number of operations in both $A\mathbf{x}$ and $\hat{A}\mathbf{x}$ and then compare.

(b) (10 points) Find an upper bound for the error $\|A\mathbf{x} - \hat{A}\mathbf{x}\|_2$ in terms of ε.

Hint: You may use $\|A - \hat{A}\|_2 \leq \sigma_{r+1}$, where $\|.\|_2$ denotes the specteral norm (or the matrix norm induced by the vector 2-norm).

4. Let us use the normalized power method to compute an approximate eigenvector

$$v_{k+1} = \frac{Av_k}{\|Av_k\|},$$

where $\|v_0\| = 1, v_0 \in \mathbb{R}^m$ and the matrix $A \in \mathbb{R}^{m\times m}$ is symmetric $A = A^T$.

(a) (7 points) Let $|\lambda_1| = |\lambda_2| > |\lambda_3| \geq \cdots \geq |\lambda_m|$ be the m eigenvalues of A with corresponding eigenvectors $q_1, q_2, \ldots, q_m$. Also let the initial vector v_0 satisfy $q_1^* v_0 \neq 0$ and $q_2^* v_0 \neq 0$.

What vector(s) will v_k converge to? Justify your answer.

(b) (7 points) Next, let $q_1^* v^0 \neq 0$ but $q_2^* v^0 = 0$.

What vector(s) will v_k converge to in exact arithmetic? Justify your answer.

What vector(s) will v_k converge to in floating point arithmetic? Justify your answer.

(c) (6 points) Derive the rate of convergence of the Rayleigh quotient $r(v_k) = v_k^T A v_k$ to λ_1 when $|\lambda_1| > |\lambda_i|$, for all $i > 1$.

5. Consider the linear system

$$\begin{bmatrix} 10^{-20} & 1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 4 \end{bmatrix}$$

(a) (4 points) Find the exact solution.

(b) (6 points) Find the solution using Gauss Elimination **without pivoting** using floating point arithmetic and the machine precision for IEEE double precision.

(c) (6 points) Find the solution using Gauss Elimination **with pivoting** using floating point arithmetic and the machine precision for IEEE double precision. *You may consider partial pivoting or scaled partial pivoting here.*

(d) (4 points) What criterion is used to implement GE **with pivoting** in practice? *You may consider partial pivoting or scaled partial pivoting here.*

6. Consider a matrix $A \in \mathbb{C}^{m,n}$.

(a) (3 points) State the singular value decomposition (SVD) of A, making sure to list all known properties of the component matrices (commonly referred to as U, Σ, and V).

(b) (3 points) State the 2-norm of a matrix, $\|A\|_2$, in terms of the singular values.

(c) (4 points) Prove your result in (b).

(d) (5 points) Define $A_k = U(\Sigma + k^{-1}I)V^*$, where U is the matrix of left singular vectors, Σ is the ordered diagonal matrix of singular values of size $m \times n$, and V is the matrix of right singular vectors. The matrix I is the identity, and k is an integer with $k \geq 1$.

Using the above parts of this question, state the value of

$$\|A - A_k\|_2.$$

(e) (5 points) Using the above parts of this question, prove that full-rank matrices are a dense subset of $\mathbb{C}^{m,n}$. That is, prove that any matrix $A \in \mathbb{C}^{m,n}$ is the limit of a sequence of matrices of full rank.

Hint: Use the result in part (d).