REAL ANALYSIS QUALIFYING EXAM

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Instructions: Complete all 8 problems to get full credit. Start each problem on a new page, number the pages, and put only your code word (not your banner ID number) on each page. Use only one side of each sheet.

Clear and concise answers with good justification will improve your score.

1. Suppose $f : \mathbb{R} \to \mathbb{R}$ is a function and that both of the functions

$$h(x) = \cos(f(x)), \quad k(x) = \sin(f(x))$$

are continuous.

(a) Give an example that shows it is possible that f is not continuous.

(b) Prove that, if we also assume $-\pi < f(x) < \pi$ for all $x \in \mathbb{R}$, then f is continuous.

2. (a) Let $f : [0,1] \to \mathbb{R}$ be a continuous function that is differentiable at some $x_0 \in (0,1)$. Prove that there exists $L > 0$ such that

$$|f(x) - f(x_0)| \le L\,|x - x_0| \quad \text{for all } x \in [0,1].$$

(b) Consider the function $f(x) = \begin{cases} x^{3/2}\sin\left(\frac{1}{x}\right) & \text{if } 0 < x \le 1 \\ 0 & \text{if } x = 0 \end{cases}.$

Prove that for every $L > 0$ there exist $x, y \in [0,1]$ such that

$$|f(x) - f(y)| > L\,|x - y|.$$

(*Hint:* analyze the derivative f'.)

3. (a) State the ε-δ definition of the uniform convergence of a sequence of functions $\{f_n\}_{n=1}^{\infty}$ to a function f on the set $\mathbb{R}$.

(b) Consider the sequence of functions given by

$$f_n(x) = \sqrt{x^2 + \frac{1}{n^2}}, \quad x \in \mathbb{R}, \quad n \in \mathbb{N}.$$

Prove that $\{f_n\}_{n=1}^{\infty}$ converges uniformly on $\mathbb{R}$.

(c) State what it means by the ε-δ definition that the sequence of functions $\{g_n\}_{n=1}^{\infty}$ does not converge to a function g uniformly on $\mathbb{R}$.

(d) Prove that the sequence of derivatives $\{f_n'\}_{n=1}^{\infty}$ of the functions defined in (b) does not converge uniformly on $\mathbb{R}$.

4. (a) Prove that, for all $n \in \mathbb{N}$, the function

$$\frac{\ln\left(e^x + n^{-1}\right)}{\left(1+x^2\right)^2}$$

is Riemann integrable on $[0, \infty)$.

(b) Evaluate

$$\lim_{n\to\infty} \int_0^\infty \frac{\ln\left(e^x + n^{-1}\right)}{\left(1+x^2\right)^2}\, dx.$$

5. Suppose $f : [-1, 1] \to \mathbb{R}$ is continuous and $f(0) = f'(0) = 0$. Prove that for every $\varepsilon > 0$ there is a polynomial $p(x)$ such that

$$\sup_{x\in[-1,1]} \left| f(x) - x^2 p(x) \right| \leq \varepsilon.$$

6. Consider the system of equations

$$\begin{aligned} u^5 + xv^2 - y + w &= 0 \\ v^5 + yu^2 - x + w &= 0 \\ w^4 + y^5 - x^4 &= 1 \end{aligned}$$

(a) Prove that there are C^1 functions $u(x, y)$, $v(x, y)$ and $w(x, y)$ defined on an open ball centered at $(1, 1)$ such that the above equations are satisfied and $u(1, 1) = 1$, $v(1, 1) = 1$, $w(1, 1) = -1$.

(b) Compute $\frac{\partial v}{\partial y}$, finding a formula in x, y, u, v, w valid in some open ball centered at $(1, 1)$.

7. (a) Let E be a nonempty open set in $\mathbb{R}^3$ and $g : E \to \mathbb{R}$ a continuous function. Denote by $B_r(P_0)$ the ball of radius r centered at P_0 and by $\partial B_r(P_0)$ its boundary surface. Prove that

$$g(P_0) = \lim_{r\to 0^+} \frac{1}{\mathrm{Vol}(B_r(P_0))} \iiint_{B_r(P_0)} g(x, y, z)\, d\,\mathrm{Vol}(x, y, z)$$

for every $P_0 \in E$.

(b) Let E be an open ball in $\mathbb{R}^3$. Let $\vec{F} : E \to \mathbb{R}^3$ be a continuously differentiable function such that $\displaystyle\iint_{\partial B_r(P_0)} \langle \vec{F}, \vec{n} \rangle\, dS = 0$ for every $P_0 \in E$ and $0 < r < \mathrm{dist}(P_0, \partial E)$.

Prove that $\mathrm{div}\, \vec{F} \equiv 0$ on E.

8. (a) Formulate the Stokes theorem.

(b) Compute the integral $\displaystyle\iint_{\partial D} \langle \mathrm{curl}\, \vec{F}, \vec{n} \rangle\, dS$,

where $D = \left\{ (x, y, z) \in \mathbb{R}^3 : \dfrac{x^2}{4} + \dfrac{y^2}{9} + \dfrac{z^2}{16} < 1 \right\}$,

$\vec{F}(x, y, z) = \Big(\cos(x \sin(z)) + y^3,\ xy,\ \cos(x \sin(ze^y)) \Big)$.

(*Hint:* apply the Stokes theorem twice to simplify the surface of integration.)