REAL ANALYSIS QUALIFYING EXAM January 14, 2020

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Instructions: Complete all 8 problems to get full credit. Start each problem on a new page, number the pages, and put only your code word (not your banner ID number) on each page. Use only one side of each sheet.

Clear and concise answers with good justification will improve your score.

- **1.** Let f be a function $f : \mathbb{R} \to \mathbb{R}$.
 - (a) Assuming f'(0) exist show that the limit

$$\lim_{h \to 0} \frac{f(h) - f(-h)}{2h} \tag{1}$$

also exists.

(b) Is it true that if the limit (1) exists then f must be continuous at 0? Prove your answer, either way.

2. Determine and prove whether the series of functions $\sum_{k=1}^{\infty} \frac{x^2}{(1+x^2)^k}$ converges on [-1,1]. If it converges, justify whether the convergence is uniform or pointwise.

- **3.** Prove that if $f : [0,1] \to \mathbb{R}$ is continuous and f(0) = 0, then there exists a sequence of polynomials $\{p_n\}_{n=1}^{\infty}$ converging to f uniformly on [0,1] and such that $p_n(0) = 0$ for every $n \in \mathbb{N}$.
- 4. Prove that the function $f(x) = \begin{cases} \sin\left(\frac{1}{x-1}\right) & \text{if } x \neq 1 \\ 0 & \text{if } x = 1 \end{cases}$ is Riemann integrable on [0, 2].

5. Let $\{f_n\}_{n=1}^{\infty}$ be a sequence of real-valued functions that is continuous on [0,1] and converges to f uniformly on [0,1]. Prove that $\lim_{n\to\infty} \int_0^{1-\frac{1}{n}} f_n(x) \, dx = \int_0^1 f(x) \, dx$.

6. Consider the function $f : \mathbb{R}^2 \to \mathbb{R}$ defined by

$$f(x,y) = \begin{cases} \frac{x^3 + x^2 y + y^3}{x^2 + y^2} & \text{if } (x,y) = (0,0) \\ 0 & \text{if } (x,y) = (0,0) \end{cases}$$

- (a) Show that f is continuous at (0, 0).
- (b) Show that both partial derivatives exist at (0,0).
- (c) Show that f is not differentiable at (0,0).

- 7. Define $f : \mathbb{R}^2 \to \mathbb{R}^2$ by $f(x, y) = (x^3 + xy + y^3, y^2 x^2)$ and let (a, b) be a point for which a > 0 and $b > -\frac{1}{3}$. Show that f is one-to-one on some open set U containing (a, b), and that that there is a differentiable function $g : f(U) \to U$ such that f(g(x, y)) = (x, y) for all $(x, y) \in U$.
- 8. (a) Formulate the Stokes theorem.

(b) Compute the integral
$$\iint_{\partial D} \langle \operatorname{curl} \vec{F}, \vec{n} \rangle \, dS,$$

where $D = \left\{ (x, y, z) \in \mathbb{R}^3 : \frac{x^2}{4} + \frac{y^2}{9} + \frac{z^2}{16} < 1 \right\},$
 $\vec{F}(x, y, z) = \left(\cos(x \sin(z)) + y^3, xy, \cos(x \sin(ze^y)) \right).$
(*Hint:* apply the Stokes theorem twice to simplify the surface of integration.)