Department of Mathematics and Statistics University of New Mexico Qualifying Exam

Real Analysis

January 2021

Instructions: Please hand in solutions to all of the 8 following problems (4 on the front page and 4 on the back page). Start each problem on a new page, number the pages, and put only your code word (not your banner ID number) on each page. Scan your exam with the solutions arranged in numerical order. Clear and concise answers with good justification will improve your score.

- 1. (a) Let f be differentiable on (a, b) and suppose that there exists $m \in \mathbb{R}$ such that $|f'(x)| \leq m$ for all $x \in (a, b)$. Prove that f is uniformly continuous on (a, b).
 - (b) Find an example of a function f that is differentiable and uniformly continuous on (0, 1), but such that f'(x) is unbounded on (0, 1).
- 2. Let $f : [a, b] \to \mathbb{R}$ be a function which is bounded and Riemann integrable on [c, b] for every $c \in (a, b)$. Prove that f is Riemann integrable on all of [a, b].
- 3. Let f be a continuous function on [0,1] such that |f(x)| < 1 for every $x \in [0,1]$.
 - (a) Prove that the numeric series $\sum_{k=1}^{\infty} (f(x))^k$ converges for every $x \in [0,1]$.
 - (b) Consider the function $F(x) := \sum_{k=1}^{\infty} (f(x))^k$. Prove that for every $\varepsilon > 0$ there exists a polynomial p_{ε} such that $\sup_{x \in [0,1]} |F(x) p_{\varepsilon}(x)| < \varepsilon$.
- 4. Let f be a Riemann integrable function on [0, 1] and define $f_n(x) := x^n f(x), n \in \mathbb{N}, x \in [0, 1].$
 - (a) Justify by stating the appropriate theorem that f_n is Riemann integrable on [0, 1] for every $n \in \mathbb{N}$.
 - (b) Find the limit of the Riemann integrals $\lim_{n\to\infty}\int_0^1 f_n(x)\,dx$ and justify each step of your solution.

- 5. Let V be open and convex in \mathbb{R}^n , and suppose $f: V \to \mathbb{R}$. Show that if each partial derivative of f exists and vanishes on all of V, then f is constant.
- 6. Suppose V is open in \mathbb{R}^n , and that $\mathbf{F} : V \to \mathbb{R}^n$ is continuously differentiable. Justify that if the derivative of \mathbf{F} is nonsingular at each $\mathbf{x} \in V$, then \mathbf{F} is an open map, that is, $\mathbf{F}(U)$ is open for each open set $U \subset V$.
- 7. Let $R \subset \mathbb{R}^2$ be defined by $R = \{(x, y) : |x| + |y| \leq 1\}$. Evaluate the integral

$$\iint\limits_R e^{x+y} \, dA$$

and justify every step of your solution.

- 8. Let $S = \left\{ (x, y, z) \in \mathbb{R}^3 : \frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1, \ z \ge 0 \right\}$ and let n_S be the outwardpointing unit normal of S. Let $A = \left\{ (x, y, z) \in \mathbb{R}^3 : \frac{x^2}{a^2} + \frac{y^2}{b^2} \le 1, \ z = 0 \right\}$ and let $n \equiv (0, 0, 1)$. Let $\mathbf{F} : \mathbb{R}^3 \to \mathbb{R}^3$ be of class C^1 . (a) Justify that $\iint_S \langle \operatorname{curl} \mathbf{F}, n_S \rangle \ d\sigma = \iint_A \langle \operatorname{curl} \mathbf{F}, n \rangle \ d\sigma$.
 - (b) Assume, in addition, that $\mathbf{F}(x, y, z) = (3y, -xz, yz^2)$. Calculate $\iint_{S} \langle \operatorname{curl} \mathbf{F}, n_S \rangle \ d\sigma$ and justify each step of your calculation.