REAL ANALYSIS QUALIFYING EXAM August 17, 2021

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Instructions: Please hand in solutions to all of the 8 following problems. Start each problem on a new page, number the pages, and put only your code word (not your banner ID number) on each page. Scan your exam with the solutions arranged in numerical order. Clear and concise answers with good justification will improve your score.

- 1. Prove that every non-empty open subset of \mathbb{R} is equal to the union of closed intervals where the number of intervals is infinite and countable.
- **2.** Prove that the function $f: [0, \infty) \to \mathbb{R}$ defined by

$$f(x) = \begin{cases} 0 & \text{if } x = 0\\ \sin(x)\cos\left(\frac{1}{x}\right) & \text{if } x > 0 \end{cases}$$

is uniformly continuous on its domain.

3. Suppose $f : [-1,1] \to \mathbb{R}$ is continuous. Prove that for every $\epsilon > 0$ there is a real polynomial p so that f(-1) = p(-1), f(1) = p(1) and

$$|f(x) - p(x)| < \epsilon$$

for all $x \in [-1, 1]$.

4. Let $f: [0,1] \to \mathbb{R}$ be a continuous function such that |f(x)| < 1 for every $x \in [0,1]$. Let $\{n_k\}_{k=1}^{\infty}$ be a strictly increasing sequence of natural numbers. Consider the function

$$F(x) := \sum_{k=1}^{\infty} (f(x))^{n_k}.$$

- (a) Prove that F is well defined on [0, 1].
- (b) Prove that F is continuous on [0, 1].
- 5. Let $f: [-1,1] \to \mathbb{R}$ be a Riemann integrable function that is continuous at x = 0. Let $\{\phi_n\}_{n=1}^{\infty}$ be a sequence of nonnegative Riemann integrable functions on [-1,1] satisfying the following two properties:
 - (i) $\int_{-1}^{1} \phi_n(t) dt = 1$ for every $n \in \mathbb{N}$;
 - (ii) For every $\delta > 0$, $\phi_n \to 0$ uniformly on $[-1, -\delta] \cup [\delta, 1]$ as $n \to \infty$.

Prove that

$$\lim_{n \to \infty} \int_{-1}^{1} f(t)\phi_n(t) \, dt = f(0).$$

6. Consider the function $f : \mathbb{R}^2 \to \mathbb{R}$ defined by

$$f(x,y) = \begin{cases} \frac{x^3 + x^2y + xy^2 + 2y^4}{x^2 + y^2} & \text{if } (x,y) \neq (0,0) \\ 0 & \text{if } (x,y) = (0,0) \end{cases}$$

- (a) Prove that f is continuous at (0, 0).
- (b) Prove that both partial derivatives exist at (0,0).
- (c) Is f is differentiable at (0, 0), and why?
- 7. Let D denote a simple region in \mathbb{R}^3 whose boundary $S = \partial D$ is a piecewise C^1 surface positively-oriented by the unit normal N. Suppose further that (0, 0, 0) in not an element of S but that (0, 0, 0) is an element of D. Let F be the vector field

$$\mathbf{F}(x,y,z) = \frac{(z-y,x-z,y-x)}{(x^2+y^2+z^2)^{\frac{3}{2}}}.$$

Prove that

$$\iint_{S} \langle \boldsymbol{F}, \boldsymbol{N} \rangle \ d\sigma = 0$$

where $d\sigma$ denotes an element of surface area.

8. Let

$$S = \{ (x, y, z) \in \mathbb{R}^3 : z = 1 - x^2 - y^2, 0 \le x^2 + y^2 \le 1 \}$$

and let \mathbf{N} be the outward-pointing unit normal of S. Let

$$\mathbf{F}(x, y, z) = \left(-y^3, \cos(y\sin(ye^z)) + x^3, \cos(x\sin(ze^y))\right).$$

Calculate $\iint_{S} \langle \operatorname{curl} \mathbf{F}, \mathbf{N} \rangle \, d\sigma$ and justify each step of your calculation.