REAL ANALYSIS QUALIFYING EXAM January 12, 2022

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Instructions: Please hand in solutions to all of the 8 following problems. Start each problem on a new page, number the pages, and put only your code word (not your banner ID number) on each page. Scan your exam with the solutions arranged in numerical order. Clear and concise answers with good justification will improve your score.

1. Prove that for every continuous function $f : \mathbb{R} \to \mathbb{R}$ there exist intervals I_1, I_2 such that

$$\sup_{x \in I_1} f(x) \le \inf_{x \in I_2} f(x)$$

- **2.** Find all function $f : \mathbb{R} \to \mathbb{R}$ that satisfy $|f(x) f(y)| \le |x y|^2$ for all $x, y \in \mathbb{R}$. Fully justify your answer.
- **3.** Prove using the contraction principle that there exists a unique function $f \in C[0, 1]$ such that $\int_0^t x f(x) dx = f(t)$ for all $t \in [0, 1]$.
- **4.** Suppose $f : [0,3] \to \mathbb{R}$ is continuous. Prove that for every $\epsilon > 0$ there is a polynomial p such that f(1) = p(1), f(2) = p(2) and for all $x \in [0,3]$,

$$|f(x) - p(x)| < \epsilon.$$

- 5. Determine and prove whether the series of functions $\sum_{k=1}^{\infty} \frac{x^2}{(1+x^2)^k}$ converges on [-1, 1]. If it converges, justify whether the convergence is uniform or pointwise.
- 6. Consider the function $\mathbf{F}: \mathbb{R}^2 \to \mathbb{R}^2$ defined by

$$\mathbf{F}(x,y) = \begin{cases} \left(\frac{x^{4/3}y^{5/3}}{x^2 + y^2}, \frac{x^{5/3}y^{4/3}}{x^2 + y^2}\right) & \text{if } (x,y) \neq (0,0)\\ (0,0) & \text{if } (x,y) = (0,0) \end{cases}$$

- (a) Prove that the partial derivatives of \mathbf{F} at (0,0) exist.
- (b) Determine and prove whether \mathbf{F} is differentiable at (0,0).

- 7. An electric charge q located at the origin produces the electric field $\mathbf{E} = \frac{q\mathbf{R}}{4\pi\varepsilon \|\mathbf{R}\|^3}$, where $\mathbf{R} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$ and ε is a physical constant.
 - (a) Let Σ_r be the positively oriented sphere with the unit normal vector **n** of radius r centered at the origin. Show that $\iint_{\Sigma_r} \langle \mathbf{E}, \mathbf{n} \rangle \, d\sigma = \frac{q}{\varepsilon}$.
 - (b) Let S be a piecewise C^1 positively oriented surface with the unit normal vector **n** such that S encloses the origin and is a boundary of a bounded domain in \mathbb{R}^3 . Prove that $\iint_{S} \langle \mathbf{E}, \mathbf{n} \rangle d\sigma = \frac{q}{\varepsilon}$.
- 8. Suppose $\mathbf{F} : \mathbb{R}^n \to \mathbb{R}^n$ is a continuously differentiable map such that the Jacobian $\det(D\mathbf{F}(x))$ is nonzero for every $x \in \mathbb{R}^n$. Prove that for every $x_0 \in \mathbb{R}^n$,

$$\lim_{r \to 0^+} \frac{\operatorname{Vol}(\mathbf{F}(B_r(x_0)))}{\operatorname{Vol}(B_r(x_0))} = |\det(D\mathbf{F}(x_0))|,$$

where $B_r(x_0)$ is the ball of radius r centered at x_0 .