Department of Mathematics and Statistics
University of New Mexico

Real Analysis **Qualifying Exam** **January 2023**

Instructions: Please hand in all of the 8 following problems (5 in the front page and 3 in the back page). Start each problem on a new page, number the pages, and put only your code word (not your banner ID number) on each page. Clear and concise answers with good justification will improve your score.

1. Show that if $\{a_k\}_{k=1}^{\infty}$ is a decreasing sequence of real numbers and $\sum_{k=1}^{\infty} a_k$ is convergent, then $\lim_{k\to\infty} k a_k = 0$.

2. Let (X, d) be a metric space and $E \subset X$ a non-empty set. Define the distance from a point $x \in X$ to the set E by
$$\rho_E(x) := \inf\{d(x, y) \,:\, y \in E\}.$$
   (a) Prove that $\rho_E(x) = 0$ if and only if x belongs to the closure of E, denoted $\overline{E}$.
   (b) Prove that ρ_E defines a uniformly continuous function on X, by showing that
$$|\rho_E(x) - \rho_E(y)| \leq d(x, y), \qquad \text{for all } x, y \in X.$$
   (c) Let A and B be disjoint nonempty closed subsets of X. Show that
$$f(x) := \frac{\rho_A(x)}{\rho_A(x) + \rho_B(x)}$$
   defines a continuous function on X whose range lies in $[0, 1]$. Moreover show that $f(x) = 0$ if and only if $x \in A$ and $f(x) = 1$ if and only if $x \in B$.
   (d) Let A and B be disjoint nonempty closed subsets of X. Show that there exists disjoint open sets $V, W \subset X$ such that $A \subset V$ and $B \subset W$.

3. Suppose $f, g : [a, b] \to \mathbb{R}$ are bounded functions and that there exists a constant $C > 0$ such that $|f(x) - f(y)| \leq C|g(x) - g(y)|$ for all $x, y \in [a, b]$. Show that if g is Riemann integrable, then f is Riemann integrable.

4. Let $f : [0, 1] \to \mathbb{R}$ be a continuous function. Show that
$$\lim_{n\to\infty} \int_0^1 n x^{n-1} f(x)\, dx = f(1).$$

5. Prove that the series $\sum_{n=1}^{\infty} 1/n^x$ converges to a differentiable function on $(1, \infty)$ and that its derivative is the sum of the derivatives of the summands.

6. Let $U \subset \mathbb{R}^n$ be an open, connected set and $f : U \to \mathbb{R}$ be a function such that $|f(x) - f(y)| \leq |x - y|^2$ for every $x, y \in U$. Prove that f is constant on U.

7. Let $U \subset \mathbb{R}^n$ be a bounded open set and denote its closure by $\overline{U}$ and its boundary by ∂U. Suppose $f : \overline{U} \to \mathbb{R}^n$ is a continuous function such that f is continuously differentiable on U and $\det(f'(x)) \neq 0$ for all $x \in U$. Prove that there exists a point $a \in \partial U$ such that $|f(x)| \leq |f(a)|$ for all $x \in \overline{U}$.

8. Let $P(x, y)$, $Q(x, y)$ be real continuously differentiable functions on $\mathbb{R}^2$. Suppose that for any circle C in $\mathbb{R}^2$, $\int_C P dx + Q dy = 0$. Prove that

$$\frac{\partial P}{\partial y}(x, y) = \frac{\partial Q}{\partial x}(x, y)$$

at each point $(x, y) \in \mathbb{R}^2$.