REAL ANALYSIS QUALIFYING EXAM

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Instructions: Please hand in solutions to all of the 8 following problems. Start each problem on a new page, number the pages, and put only your code word (not your banner ID number) on each page. Clear and concise answers with good justification will improve your score.

1. Let $f : [0,1] \to \mathbb{R}$ be a continuously differentiable function. Let S_k denote the Riemann sum for f over the partition of step size $\frac{1}{k}$, $k \in \mathbb{N}$. Prove that there exists a constant C such that

$$\left| S_k - \int_0^1 f(x)\,dx \right| < \frac{C}{k}$$

for all $k \in \mathbb{N}$.

2. Prove that

$$f_n(x) = n \sin(x/n)$$

converges uniformly to $f(x) = x$ on every bounded interval.

3. Suppose $f : [-1,1] \to \mathbb{R}$ is twice differentiable and is an odd function. Prove that there is a sequence of polynomials p_n such that

$$|f(x) - p_n(x)| \leq \frac{|x|}{n}$$

for all $n \in \mathbb{N}$ and all x in $[-1,1]$.

4. Let $f : \mathbb{R} \to \mathbb{R}$ be define by $f(0) = 0$ and

$$f(x) = x \sin^2(1/x)$$

for $x \neq 0$.

(a) Is f continuous on $\mathbb{R}$? Fully justify your answer.

(b) Is f uniformly continuous on $\mathbb{R}$? Fully justify your answer.

(c) Is f differentiable on $\mathbb{R}$? Fully justify your answer.

5. (a) Prove that if $F : \mathbb{R}^2 \to \mathbb{R}^2$ is continuous at $(0,0)$ and satisfies

$$\langle F(x,y),(x,y)\rangle = 0$$

for all $x, y \in \mathbb{R}$, then $F(0,0) = (0,0)$.

Here and below $\langle \mathbf{a}, \mathbf{b}\rangle$ denotes the canonical inner product of vectors $\mathbf{a}, \mathbf{b} \in \mathbb{R}^n$.

(b) Prove that if $F : \mathbb{R}^n \to \mathbb{R}^m$ is differentiable at $x_0 \in \mathbb{R}^n$, then it is continuous at x_0.

6. Consider the vector equation

$$\mathbf{x}^{(2)} + e^{\langle \mathbf{x},\mathbf{y}\rangle}\mathbf{y} = \mathbf{0}$$

where $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$ and $\mathbf{x}^{(2)} := (x_1^2, x_2^2, \ldots, x_n^2)$.

(a) Prove that there exist $\varepsilon > 0$ and a continuously differentiable function $g : B_\varepsilon(\mathbf{0}) \to \mathbb{R}^n$ such that $\mathbf{y} = g(\mathbf{x})$ is a solution to the given equation for all $\mathbf{x} \in B_\varepsilon(\mathbf{0})$.

Here and below $B_\varepsilon(\mathbf{0})$ denotes the ball in $\mathbb{R}^n$ centered at $\mathbf{0}$ and of radius ε.

(b) Compute $Dg(\mathbf{0})$ and justify each step of your calculation.

(c) Prove that for every $\varepsilon > 0$, the given equation cannot have a solution $\mathbf{x} = h(\mathbf{y})$ defined for all $\mathbf{y} \in B_\varepsilon(\mathbf{0})$.

7. Suppose $F : \mathbb{R}^n \to \mathbb{R}^m$ is a continuously differentiable function and let $a \in \mathbb{R}^n$. Prove that if $F'(a)$ has rank m, then there exists a neighborhood U of a in which F' has rank at least m throughout U.

(Hint: recall that an $m \times n$ matrix has rank m if and only if there exists an $m \times m$ minor with nonvanishing determinant.)

8. (a) Prove that, for all $n \in \mathbb{N}$, the function

$$\frac{x^{\frac{n-1}{n}}}{(1+x^2)^2}$$

is Riemmann integrable on $[1, \infty)$.

(b) Evaluate

$$\lim_{n\to\infty} \int_1^\infty \frac{x^{\frac{n-1}{n}}}{(1+x^2)^2}\, dx$$

and justify each step of your solution.