REACH SET COMPUTATION FOR LINEAR VECTOR FIELDS USING QUANTIFIER ELIMINATION (EXTENDED ABSTRACT)

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1. Introduction

Hybrid systems [12] combine discrete event systems with differential equations. One of the most important problems for hybrid systems is the *reachability problem* which asks whether some region of the state space is reachable from an initial region. The important issue for computational algorithms for hybrid systems is *decidability*. A survey of decidable and undecidable classes of hybrid systems with continuous dynamics of the form $\dot{x} \in [a, b]$ can be found in [6] and the references therein.

Recently, in [8, 9], we have shown new decidability results involving hybrid systems with linear vector fields of the form $\dot{x} = Ax + Bu$. These results are significant given the wide applicability of linear systems in control theory. In order to achieve them we have heavily relied on two key concepts from model theory, namely, *o-minimality* [11] and *quantifier elimination* [2, 10, 13, 4].

In our decidability results, quantifier elimination plays an important role when dealing with the continuous dynamics. For this reason, in this paper we abstract away the discrete structure of hybrid systems and restric our study to the linear vector fields. Our approach consists in characterizing the set of reachable states as a predicate in the theory of the ordered field of real numbers. A quantifier-free characterization can then be obtained by quantifier elimination using tools such as Redlog [5] and Qepcad [4]. In the full paper we illustrate the results with several examples solved using Redlog.

2. Reachability problem for linear vector fields

Let $F(x) = Ax + Bu$ with $x \in \mathbb{R}^n$, $A \in \mathbb{Q}^{n\times n}$, $u : \mathbb{R} \to \mathbb{R}^m$ and $B \in \mathbb{Q}^{n\times m}$. Let

$$\Phi(s,t,x) = e^{A(t-s)}x + \int_s^t e^{A(t-\tau)}Bu(\tau)\, d\tau \tag{2.1}$$

where

$$e^{At} = \sum_{k=0}^{\infty} \frac{t^k}{k!}A^k \tag{2.2}$$

Let $Y \subseteq \mathbb{R}^n$. We denote by $Pre(Y)$ the following set:

$$Pre(Y) \quad = \quad \{x \in \mathbb{R}^n \mid \exists y\ \exists t\ \exists s : y \in Y \wedge s \geq 0 \wedge t \geq s \wedge \Phi(s,t,x) = y\}$$

$Pre(Y)$ is the so-called *backward reachability set* of Y. that is, $Pre(Y)$ contains all the points from where it is possible to reach a point in Y following the flow of the vector field F. The *forward reachability set* can be defined analogously.

Now, let $\mathcal{R}$ be the theory of the ordered field of real numbers and $\mathcal{R}_{\text{exp}}$ and $\mathcal{R}_{\text{sin}}$ respectively be its extensions with the exponentiation function e^t and the trigonometric functions $\sin(t)$ and $\cos(t)$.

Clearly, $Pre(Y)$ is definable in $\mathcal{R}_{\text{exp}}$. Based on the eigenstructure of A, we shall identify several cases for which $Pre(Y)$ is actually definable in $\mathcal{R}$. This result implies that, in those cases, $Pre(Y)$ is indeed computable and one has a decision procedure for solving the reachability problem based on quantifier elimination.

3. Linear vector fields without inputs

In this section we consider vector fields of the form $F(x) = Ax$. Let $Y \subseteq \mathbb{R}^n$ be definable in $\mathcal{R}$, that is, $Y = \{y \in \mathbb{R}^n \mid P(y)\}$ for some predicate P of $\mathcal{R}$. Then,

$$Pre(Y) \;=\; \{x \in \mathbb{R}^n \mid \exists y\, \exists t : P(y) \wedge t \geq 0 \wedge x = e^{-tA}y\}$$

Let $\eta(x)$ in $\mathcal{R}_{\text{exp}}$ be defined as follows:

$$\eta(x) \;\triangleq\; \exists y\, \exists t : P(y) \wedge t \geq 0 \wedge x = e^{-tA}y$$

The goal is to transform the formula $\eta(x)$ to an equivalent formula in $\mathcal{R}$.

3.1. Nilpotent matrices.

Let A such that $A^n = 0$. Then

$$e^{-tA} = \sum_{k=0}^{n-1} (-1)^k \frac{t^k}{k!} A^k \tag{3.1}$$

Thus, the formula $\eta(x)$ can be rewritten as follows:

$$\eta(x) \;\triangleq\; \exists y\, \exists t : P(y) \wedge t \geq 0 \wedge x = \sum_{k=0}^{n-1} (-1)^k \frac{t^k}{k!} A^k y$$

Proposition 3.1. *Let $F(x) = Ax$ with $A \in \mathbb{Q}^{n\times n}$ a nilpotent matrix, and $Y \subseteq \mathbb{R}^n$ definable in $\mathcal{R}$. Then $Pre(Y)$ is definable in $\mathcal{R}$.*

3.2. Diagonalizable matrices with rational eigenvalues.

In this case we can write $A = TDT^{-1}$ where D is a diagonal matrix with the eigenvalues of A along the diagonal and both T and T^{-1} have rational entries. Then

$$e^{-tA} = e^{-tTDT^{-1}} = T \begin{bmatrix} e^{-t\lambda_1} & & \\ & \ddots & \\ & & e^{-t\lambda_n} \end{bmatrix} T^{-1} = [f_{ij}(t)] \tag{3.2}$$

where $f_{ij}(t) = \sum_{k=1}^{n} a_{ijk} e^{-\lambda_k t}$ with $a_{ijk} \in \mathbb{Q}$, and $\{\lambda_k\}$ are the eigenvalues of A. $x = e^{-tA}y$ can be written component-wise as follows:

$$x_i = \sum_{j=1}^{n} \left(\sum_{k=1}^{n} a_{ijk} e^{-\lambda_k t} \right) y_j = \sum_{k=1}^{n} \left(\sum_{j=1}^{n} a_{ijk} y_j \right) e^{-\lambda_k t} = \sum_{k=1}^{n} \psi_{ik}(y) e^{-\lambda_k t}$$

Therefore,

$$\eta(x) \equiv \exists y\, \exists t : P(y) \wedge t \geq 0 \wedge \bigwedge_{i=1}^{n} x_i = \sum_{k=1}^{n} \psi_{ik}(y) e^{-\lambda_k t}$$

For each $k = 1, \ldots, n$ let d_k denote the denominator of λ_k and let $d_0 = \prod d_k$. We assume that the λ_k are in reduced form, with positive denominators. Then $d_0 > 0$ and for each $k = 1, \ldots, n$ we write $r_k = \lambda_k d_0$. Then by using the change of variable $t = d_0 s$ we have:

$$\eta(x) \equiv \exists y\, \exists s : P(y) \wedge s \geq 0 \wedge \bigwedge_{i=1}^{n} x_i = \sum_{k=1}^{n} \psi_{ik}(y)\; e^{-r_k s}$$

We subsitute e^t by z and we obtain:

$$\eta(x) \equiv \exists y\, \exists z : P(y) \wedge 0 < z \leq 1 \wedge \bigwedge_{i=1}^{n} x_i = \sum_{k=1}^{n} \psi_{ik}(y)\; z^{r_k}$$

The third step eliminates negative polynomial powers. It consists of grouping the indices $1, \ldots, n$ according to the sign of the corresponding eigenvalue. Let $I^+ = \{k \mid r_k > 0\}$, $I^- = \{k \mid r_k < 0\}$, and $I^0 = \{k \mid r_k = 0\}$. Then,

$$\begin{aligned} \eta(x) \equiv \exists y\, \exists w_1\, \exists w_2 :& \\ & P(y) \wedge w_1 > 0 \wedge w_2 > 0 \wedge w_1 w_2 = 1 \\ & \wedge \bigwedge_{i=1}^{n} x_i = \sum_{k \in I^+} \psi_{ik}(y)\; w_1^{r_k} + \sum_{k \in I^-} \psi_{ik}(y)\; w_2^{-r_k} + \sum_{k \in I^0} \psi_{ik}(y) \end{aligned}$$

Proposition 3.2. *Let $F(x) = Ax$ with $A \in \mathbb{Q}^{n \times n}$ a diagonalizable matrix with rational eigenvalues, and $Y \subseteq \mathbb{R}^n$ definable in $\mathcal{R}$. Then $Pre(Y)$ is definable in $\mathcal{R}$.*

3.3. **Pure imaginary eigenvalues.** In this case the matrix A is similar to a matrix in a special block-diagonal form, a real Jordan form. Second, That is, there exist D and T such that $A = TDT^{-1}$, T invertible, and D is block diagonal with blocks $D_1, \ldots, D_m$ ($n = 2m$) of the form

$$D_k = \begin{bmatrix} 0 & \beta_k \\ -\beta_k & 0 \end{bmatrix}$$

where $\beta_k \in \mathbb{Q}$ is the imaginary part of an eigenvalue of A. Then,

$$e^{-tA} = e^{-tTDT^{-1}} = T \begin{bmatrix} e^{-tD_1} & & \\ & \ddots & \\ & & e^{-tD_m} \end{bmatrix} T^{-1}$$

where

$$e^{-tD_k} = \begin{bmatrix} \cos(\beta_k t) & -\sin(\beta_k t) \\ \sin(bt) & \cos(bt) \end{bmatrix}.$$

Then, $e^{-tA} = [f_{ij}(t)]$ with $f_{ij}(t) = \sum_{k=1}^{n} a_{ijk}\cos(\beta_k t) + b_{ijk}\sin(\beta_k t)$ with $a_{ijk}, b_{ijk}, \beta_k \in \mathbb{Q}$. The formula $x = e^{-tA}y$ can be written component-wise as follows:

$$\begin{aligned} x_i &= \sum_{j=1}^{n}\left(\sum_{k=1}^{n}(a_{ijk}\cos(\beta_k t) + b_{ijk}\sin(\beta_k t)\right) y_j \\ &= \sum_{k=1}^{n}\left(\sum_{j=1}^{n} a_{ijk}y_j\right)\cos(\beta_k t) + \sum_{k=1}^{n}\left(\sum_{j=1}^{n} b_{ijk}y_j\right)\sin(\beta_k t) \\ &= \sum_{k=1}^{n}\psi_{ik}^{a}(y)\cos(\beta_k t) + \psi_{ik}^{b}(y)\sin(\beta_k t) \end{aligned}$$

Therefore, $\eta(x)$ can be rewritten as follows:

$$\eta(x) \quad \triangleq \quad \exists y\, \exists t : P(y) \wedge t \geq 0 \wedge \bigwedge_{i=1}^{n} x_i = \sum_{k=1}^{n}\psi_{ik}^{a}(y)\cos(\beta_k t) + \psi_{ik}^{b}(y)\sin(\beta_k t)$$

For each $k = 1, \ldots, n$ let d_k denote the denominator of β_k and let $d_0 = \prod d_k$. We assume that the β_k are in reduced form, with positive denominators. Then $d_0 > 0$ and for each $k = 1, \ldots, n$ we write $r_k = \beta_k d_0$. Then by using the change of variable $t = d_0 s$ we have:

$$\eta(x) \quad \equiv \quad \exists y\, \exists s : P(y) \wedge s \geq 0 \wedge \bigwedge_{i=1}^{n} x_i = \sum_{k=1}^{n}\psi_{ik}^{a}(y)\cos(r_k s) + \psi_{ik}^{b}(y)\sin(r_k s)$$

Proposition 3.3. *For each $m \geq 1$ there exist homogeneous polynomials $f_m(x, y)$ and $g_m(x, y)$ of degree m such that $\cos(ms) = f_m(\cos s, \sin s)$ and $\sin(ms) = g_m(\cos s, \sin s)$.*

Using the proposition above we obtain:

$$\begin{aligned} \eta(x) \quad \equiv \quad & \exists y\, \exists s : P(y) \wedge s \geq 0 \wedge \\ & \bigwedge_{i=1}^{n} x_i = \sum_{k=1}^{n}\psi_{ik}^{a}(y) f_{|r_k|}(\cos s, \mathrm{sign}(r_k)\sin s) + \psi_{ik}^{b}(y) g_{|r_k|}(\cos s, \mathrm{sign}(r_k)\sin s). \end{aligned}$$

Due to the periodicity of both sin and cos we have that:

$$\begin{aligned} \eta(x) \quad \equiv \quad & \exists y\, \exists s : P(y) \wedge 0 \leq s \leq 2\pi \wedge \\ & \bigwedge_{i=1}^{n} x_i = \sum_{k=1}^{n}\psi_{ik}^{a}(y) f_{|r_k|}(\cos s, \mathrm{sign}(r_k)\sin s) + \psi_{ik}^{b}(y) g_{|r_k|}(\cos s, \mathrm{sign}(r_k)\sin s). \end{aligned}$$

Using the trigonometric equation $\cos^2 s + \sin^2 s = 1$ and the change of variables $z_1 = \cos s$ and $z_2 = \sin s$, we obtain:

$$\begin{aligned} \eta(x) \quad \equiv \quad & \exists y\, \exists z_1\, \exists z_2 : P(y) \wedge z_1^2 + z_2^2 = 1 \wedge \\ & \bigwedge_{i=1}^{n} x_i = \sum_{k=1}^{n}\psi_{ik}^{a}(y) f_{|r_k|}(z_1, \mathrm{sign}(r_k) z_2) + \psi_{ik}^{b}(y) g_{|r_k|}(z_1, \mathrm{sign}(r_k) z_2). \end{aligned}$$

Proposition 3.4. *Let $F(x) = Ax$ with $A \in \mathbb{Q}^{n\times n}$ having pure imaginary eigenvalues of the form* $\mathbf{i}r$ *with $r \in \mathbb{Q}$, and $Y \subseteq \mathbb{R}^n$ definable in $\mathcal{R}$. Then $Pre(Y)$ is definable in $\mathcal{R}$.*

Theorem 3.5. *Let $F(x) = Ax$ be a linear vector field and $A \in \mathbb{Q}^{n\times n}$ where*

- *A is nilpotent or*
- *A is diagonalizable with rational eigenvalues or*
- *A has pure imaginary eigenvalues of the form $\mathbf{i}r$, $r \in \mathbb{Q}$.*

Then the reachability problem for F is decidable.

4. Linear vector fields with inputs

In this section we consider vector fields of the form $F(x) = Ax + Bu$. The idea here consists in finding cases in which the result of integrating

$$\int_s^t e^{A(t-\tau)} Bu(\tau)\, d\tau$$

yields a formula $\Psi(s,t,x)$ such that $\Phi(s,t,x) = e^{At} + \Psi(s,t,x)$ can be characterized by a formula in $\mathcal{R}$.

4.1. **Nilpotent matrices.** We have shown that if A is nilpotent then each entry of e^{At} is a polynomial in t. It follows that, if $u(t)$ is a vector of polynomials, $\Phi(s,t,x)$ is, after integration, a vector of polynomials (whihch can be computed). Therefore, $Pre(Y)$ can be characterized by a formula in $\mathcal{R}$.

4.2. **Diagonalizable matrices with rational eigenvalues.** Let A be diagonalizable with rational eigenvalues and $u(t)$ be a vector of exponential functions of the form $e^{\mu_j}t$ with $\mu_j \in \mathbb{Q}$ not an eigenvalue of A.

We can show that each entry of e^{At} is a linear combination of the functions $e^{\lambda_i t}$ with λ_i an eigenvalue of A. Then, the entries of $e^{-A\tau}Bu(\tau)$ are linear combinations of $e^{(\mu_j-\lambda_i)t}$. By the assumption $\mu_j - \lambda_i \neq 0$ it results (after integration) that the entries of $\Phi(s,t,x)$ are also linear combinations of $e^{\lambda_i t}$, $e^{(\mu_j-\lambda_i)t}$, $e^{\lambda_i s}$, and $e^{(\mu_j-\lambda_i)s}$, and therefore definable in $\mathcal{R}_{\exp}$.

We obtain that $\Phi(x,t,x) = y$ can be written component-wise in such a way that for each $1 \leq i \leq n$ we have that:

$$y_i = \sum_k \psi_{ik}(x) e^{\mu_k t} e^{-\mu_k s} + \sum_l a_{il}(e^{\nu_l t} - e^{\nu_l s})$$

for some linear functions ψ_{ik} of x and rational constants a_{il}, μ_k, and ν_l.

Now, let d_k and c_l be the least common denominators of the μ_k and ν_l (both assumed positive), respectively. Set $d_0 = \prod d_k$, $c_0 = \prod c_l$, $p_k = \mu_k d_0$, and $q_l = \nu_l c_0$. Then,

$$\eta(x) \;\equiv\; \exists y\, \exists \tilde{t}\, \exists \tilde{s} : P(y) \wedge d_0 \tilde{t} \geq c_0 \tilde{s} \wedge y_i = \sum_k \psi_{ik}(x) e^{p_k \tilde{t}} e^{-p_k \tilde{s}} + \sum_l a_{il}(e^{q_l \tilde{t}} - e^{q_l \tilde{s}}).$$

Using the substitutions $z = e^{\tilde{t}}$, $w = e^{-\tilde{s}}$, we obtain:

$$\eta(x) \equiv \exists y\, \exists z\, \exists w :$$
$$P(y) \wedge z > 0 \wedge w > 0 \wedge w^{c_0} z^{d_0} > 1 \wedge y_i = \sum_k \psi_{ik}(x) z^{p_k} w^{p_k} + \sum_l a_{il}(z^{q_l} - w^{-q_l}).$$

which can be rewritten in $\mathcal{R}$ by multiplying through by $\prod w^{q_l}$.

4.3. **Pure imaginary eigenvalues.** Let A be a matrix with pure imaginary eigenvalues of the form $\mathbf{i}\omega_k$, $\omega_k \in \mathbb{Q}$, and $u(t)$ be a vector such that each entry is of the form $\sin(\alpha_j t)$ or $\cos(\alpha_j t)$ with $\alpha_j \in \mathbb{Q}$ and $\alpha_j \neq \pm\omega_k$ for all j, k.

We have already shown that the entries of e^{At} are linear combinations of $\cos(\omega_k t)$ and $\sin(\omega_k t)$. Because of the form of the input $u(t)$, the entries of $e^{-A\tau}Bu(\tau)$ are linear combinations of the products $\cos(\alpha_j t)\cos(\omega_k t)$, $\cos(\alpha_j t)\sin(\omega_k t)$, $\sin(\alpha_j t)\cos(\omega_k t)$, and $\sin(\alpha_j t)\sin(\omega_k t)$. Using standard product formulas these can be rewritten as linear combinations of $\sin((\alpha_j \pm \omega_k)t)$ and $\cos((\alpha_j \pm \omega_k)t)$. Since $\alpha_j \neq \pm\omega_k$, integration again results in linear combinations of $\sin((\alpha_j \pm \omega_k)t)$ and $\cos((\alpha_j \pm \omega_k)t)$ and therefore $\Phi(s, t, x)$ is definable in $\mathcal{R}_{\sin}$.

As in the previous section, we can show that the substitutions $z = \cos(t)$, $w = \sin(t)$ and the identity $z^2 + w^2 = 1$ (and similarly for the variable s) lead to the equivalent formula in $\mathcal{R}$.

Remark 4.1. The above results remain valid if the inputs are allowed to be linear combinations of the functions of the corresponding type: exponentials in case of real eigenvalues, and sinusoidal in the case of imaginary eigenvalues. In all cases the same “resonance” restrictions apply on the parameters μ and α.

5. Conclusions

In [8, 9] we use the results presented here and those in [7] to show that the reachability problem is decidable for hybrid systems. These results provide a natural and unified platform for pursuing further research along this direction.

Issues for further research include the study differential inclusions of the form $\dot{x} = Ax + Bu, u \in U$. This requires certain monotonicity conditions on the solution integral

Another direction of research includes the implementation of a computational tool whose kernel will be a quantifier elimination engine.

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