

Symbolic and Numeric Computation of the Barnes Function

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Abstract

The Barnes G function, defined by a functional equation as a generalization of the Euler gamma function, is used in many applications of pure and applied mathematics, and theoretical physics. The theory of the Barnes function has been related to certain spectral functions in mathematical physics, to the study of functional determinants of Laplacians of the n -sphere, to the Hecke L -functions, and to the Selberg zeta function. There is a wide class of definite integrals and infinite sums appearing in statistical physics (the Potts model) and lattice theory which can be computed by means of the G function. This talk presents new integral representations, asymptotic series and some special values of the Barnes function. An explicit representation by means of the Hurwitz zeta function and its relation to the determinants of Laplacians are also discussed. Finally, I propose an efficient numeric procedure for evaluating the G function.

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Preamble

In the summer of 2000 there was a conference (on special functions) in Arizona, where Prof. Richard Askey (*q*-special functions guru) gave an outlook for the new flowers in the special function "garden" (digital, as a matter of fact) that have yet to be discovered or fully investigated. The Barnes function was on his list.

Historical Sketch

The multiple Barnes function, first introduced and studied by Barnes (same Barnes as in Mellin-Barnes contour integrals) in around 1900s, is defined as a generalization of the classical Euler gamma function, by the following recurrence-functional equation:

$$\begin{aligned} G_{n+1}(z+1) &= \frac{G_{n+1}(z)}{G_n(z)}, \quad z \in C, \quad n \in \mathbb{N} \\ G_1(z) &= \frac{1}{\Gamma(z)} \\ G_n(1) &= 1 \end{aligned} \tag{1}$$

where $\Gamma(z)$ is the Euler gamma function. Surely, this system has no unique solution (recall the Bohr-Mollerup theorem for the gamma function). We have to add the condition of convexity. Originally, the G function appeared 40 years earlier (but in a different form) in Kinkelin and Glaisher papers on specific classes of numeric products (which I will discuss in details in the next section).

For those who know: the Barnes G -function has nothing to do with the Meijer G -function!!

To get a sense of this function, consider a particular case $n = 2$:

$$G_2(z+1) = \Gamma(z) G_2(z)$$

and compare it with the functional equation for the gamma function

$$\Gamma(z+1) = z \Gamma(z)$$

As $\Gamma(n+1) = n!$ is the product of natural numbers, $G_2(n+1)$ is the product of factorials

$$G_2(n+1) = \prod_{k=1}^{n-1} k!$$

The Barnes function (or functions -?) is not listed in the tables (Abramowitz and Stegun, of course, the legendary handbook which appeared three years after I was born) of the most well-known special functions, however it was cited in the exercises by Whittaker and Watson (p.264), and in entries 6.441, 8.333 by Gradshteyn and Ryzhik.

The theory of the G_n function remains the active topic of research. Vigneras and Voros showed the connection of G_n to the Selberg zeta function. Vardi, and Quine and Choi computed the functional determinants of Laplacians of the n -sphere in terms of the Barnes function. Weisberger and Sarnak demonstrated the relation of determinants to superstring theory. Gosper (macsyma man) worked out a few nice definite integrals related to the G function.

Generalized Glaisher's constant

First Kinkelin and Glaisher (1860), then Alexeiewsky(1894), Bendersky (1933), Ramanujan, and later MacLeod (1982) tackled the problem of the asymptotic expansion when $n \rightarrow \infty$ of the following product

$$1^{1^p} 2^{2^p} \dots n^{n^p} \quad (2)$$

which could be considered as a natural generalization of the Stirling formula for $n!$. To see how this product is related to the Barnes function, we focus on the simpler form when $p = 1$:

$$\begin{array}{rcl}
 1^1 2^2 3^3 \dots n^n & = & \\
 1 * 2 * 3 * \dots * n & \rightarrow & n! \\
 2 * 3 * \dots * n & \rightarrow & \frac{n!}{1!} \\
 & & \frac{n!^n}{1! 2! \dots (n-1)!} \\
 3 * \dots * n & \rightarrow & \frac{n!}{2!} \\
 \dots & & \\
 n & \rightarrow & \frac{n!}{(n-1)!}
 \end{array}$$

This simple rearrangement gives

$$1^1 2^2 \dots n^n = \frac{n!^n}{G_2(n+1)} \quad (3)$$

or in alternative form

$$\sum_{m=1}^n m \log m = n \log n! - \log G_2(n+1) \quad (4)$$

Similar regrouping techniques applied to the product (2) with $p = 2$, lead to the following identity

$$\sum_{m=1}^n m^2 \log m = n^2 \log n! - (2n-1) \log G_2(n+1) - 2 \log G_3(n+1) \quad (5)$$

Bendersky generalized all previous results on the asymptotic expansion of the product (2) and stated that

$$\log A_p = \lim_{n \rightarrow \infty} \left(\sum_{m=1}^n m^p \log m - R(n, p) \right) \quad (6)$$

is a constant (free of n , but depending on a parameter p), where

$$R(n, p) = \left(\frac{n^{p+1}}{p+1} + \frac{n^p}{2} + \frac{B_{p+1}}{p+1} \right) \log n - \frac{n^{p+1}}{(p+1)^2} + p! \sum_{j=1}^{p-1} \frac{\log n + H_p - H_{p-j}}{(j+1)! (p-j)!} n^{p-j} B_{j+1}$$

where H_p and B_p are harmonic and Bernoulli numbers respectively.

Letter A in the left hand side of (6) is a historic artifact and has nothing to do with my last name

Bendersky was attempting to find a closed form representation of constants A_k in terms of elementary functions. The first constant is indeed "less" transcendental:

$$\log A_0 = \log \sqrt{2\pi} = -\zeta'(0)$$

Though the second constant is not so trivial:

$$\log A_1 = \frac{1}{12} - \zeta'(-1)$$

and is known as the Glaisher-Kinkelin constant A (no subscript, $A = A_1$). In above formulas ζ is the Riemann zeta function. Bendersky was not able to identify all other constants. However, as it turns out, the general form is simple and quite nice:

$$\log A_p = \frac{H_p B_{p+1}}{p+1} - \zeta'(-p) \quad (7)$$

I named A_p generalized Glaisher's constant.

Remark. According to Bruce Berndt (see Entry 27(a) and (b)), in the first notebook Ramanujan derived a full asymptotic expansion (6), which includes an explicit form for the constants (7) as well.

Remark. In a view of formulas (4) and (5), using Bendersky's result (6) along with the formula (7), we can derive an asymptotic expansion of the Barnes function.

Asymptotic Expansions

I don't know if there is an easier way, but I always work out asymptotics through the suitable integral representations. In this case, by analogy with the gamma function, they are the Binet integrals.

Proposition. For $R(z) > 0$:

$$\log G_2(z+1) = \frac{z^2}{2} (\log z - H_2) - (z\zeta'(0) - \zeta'(-1)) - \int_0^\infty \frac{x \log(x^2 + z^2)}{e^{2\pi x} - 1} dx$$

$$2 \log G_3(z+1) = -\frac{z^3}{3} (\log z - H_3) + \log G_2(z+1) + (z^2 \zeta'(0) - 2z\zeta'(-1) + \zeta'(-2)) + 2 \int_0^\infty \frac{x^2 \arctan(x/z)}{e^{2\pi x} - 1} dx$$

At first glance, this is quite surprising to see that Barnes functions have different integral representations. On the other hand, the above formulas have an inner order. To see the structure we need to derive a few more formulas. Deriving each of them is quite tedious work, but thanks to Tony Hearn (am I too bias?), there are computer algebra systems. In the spirit of things, I must be even and refer to all (just two?) of the competing religions or to none.

$$\begin{aligned}
3! \log G_4(z+1) &= \frac{z^4}{4} (\log(z) - H_4) - \log G_2(z+1) + 6 \log G_3(z+1) - \\
&\quad (z^3 \zeta'(0) - 3 z^2 \zeta'(-1) + 3 z \zeta'(-2) - \zeta'(-3)) + \int_0^\infty \frac{x^3 \log(x^2 + z^2)}{e^{2\pi x} - 1} dx \\
4! \log G_5(z+1) &= -\frac{z^5}{5} (\log(z) - H_5) + \log G_2(z+1) - 14 \log G_3(z+1) + 36 \log G_4(z+1) + \\
&\quad (z^4 \zeta'(0) - 4 z^3 \zeta'(-1) + 6 z^2 \zeta'(-2) - 4 z \zeta'(-3) + \zeta'(-4)) - 2 \int_0^\infty \frac{x^4 \arctan(x/z)}{e^{2\pi x} - 1} dx
\end{aligned}$$

The inner structure is almost clear, except the sequences $\{1, 6\}$ and $\{1, 14, 36\}$. Having only three numbers, it's hard to guess the general form. So we look at the formula for $\log G_6(z+1)$:

$$5! \log G_6(z+1) \rightarrow -\log G_2(z+1) + 30 \log G_3(z+1) - 150 \log G_4(z+1) + 240 \log G_5(z+1) + < \text{skip} >$$

Now we have

$$\begin{aligned}
&1, 6, \\
&1, 14, 36, \\
&1, 30, 150, 240
\end{aligned}$$

If I was Ramanujan I'd immediately shout: "These are the Stirling numbers of the second kind multiplied by a factorial". Fortunately, we have Sloane's on-line encyclopedia of sequences!

Proposition. For $R(z) > 0$:

$$\begin{aligned}
(2n)! \log G_{2n+1}(z+1) &= -\frac{z^{2n+1} (\log(z) - H_{2n+1})}{2n+1} + \sum_{k=0}^{2n} (-1)^k \left\{ \begin{matrix} 2n \\ k \end{matrix} \right\} \zeta'(-k) z^{2n-k} - \\
&\quad \sum_{k=1}^{2n-1} (-1)^k k! \left\{ \begin{matrix} 2n \\ k \end{matrix} \right\} \log G_{k+1}(z+1) - 2(-1)^n \int_0^\infty \frac{x^{2n} \arctan(\frac{x}{z})}{e^{2\pi x} - 1} dx
\end{aligned} \tag{8}$$

$$\begin{aligned}
(2n-1)! \log G_{2n}(z+1) &= \frac{z^{2n} (\log(z) - H_{2n})}{2n} - \sum_{k=0}^{2n-1} (-1)^k \left\{ \begin{matrix} 2n-1 \\ k \end{matrix} \right\} \zeta'(-k) z^{2n-k-1} + \\
&\quad \sum_{k=1}^{2n-2} (-1)^k k! \left\{ \begin{matrix} 2n-1 \\ k \end{matrix} \right\} \log G_{k+1}(z+1) + (-1)^n \int_0^\infty \frac{x^{2n-1} \log(z^2 + x^2)}{e^{2\pi x} - 1} dx
\end{aligned} \tag{9}$$

where $\left\{ \begin{matrix} n \\ k \end{matrix} \right\}$ are Stirling numbers of the second kind. Asymptotic expansions of the Barnes functions at infinity follow from (8) and (9) by expanding $\log(x^2 + 1)$ and $\arctan(x)$ into the Taylor series, and then performing term-by-term integration. Here is the simplest formula

Proposition. For $R(z) > 0$:

$$\log G_2(z+1) = \frac{z^2}{2} \left(\log z - \frac{3}{2} \right) - \frac{\log z}{12} - z \zeta'(0) + \zeta'(-1) - \sum_{k=1}^n \frac{B_{2k+2}}{(4k(k+1)) z^{2k}} + O\left(\frac{1}{z^{2n+2}}\right)$$

In the era of computer algebra, there is no need to derive explicit formulas anymore, and so I won't. One can ask, what about the proof?

Special Cases

Since the Barnes function is the generalization of the factorial of the gamma function, it is quite surprising to anticipate that the G_n would have closed form representations for particular arguments. As a matter of fact, the Barnes function for rational arguments can be expressed in terms of Glaisher's, as well as other, known constants. Consider the integral

$$\int_0^z \log \Gamma(x) dx = \frac{z(1-z)}{2} + z \log \sqrt{2\pi} - (1-z) \log \Gamma(z) - \log G_2(z) \quad (10)$$

(originally evaluated by Gosper and independently by me), the source of all closed form representations. The simplest one (but not-trivial) is due to Barnes:

$$\log G_2\left(\frac{1}{2}\right) = \frac{1}{8} + \frac{\log 2}{24} - \frac{\log \pi}{4} - \frac{3}{2} \log A \quad (11)$$

The G_2 function, similar to the Gamma function, poses the duplication formula, which, applying to (11), gives us (due to Srivastava and Choi):

$$\begin{aligned} \log G_2\left(\frac{1}{4}\right) &= \frac{3}{32} - \frac{G}{4\pi} - \frac{3}{4} \log \Gamma\left(\frac{1}{4}\right) - \frac{9}{8} \log A \\ \log G_2\left(\frac{3}{4}\right) &= \log G_2\left(\frac{1}{4}\right) + \frac{G}{2\pi} - \frac{\log 2}{8} - \frac{\log \pi}{4} + \log \Gamma\left(\frac{1}{4}\right) \end{aligned}$$

I have derived a few other special cases for $z = \frac{1}{6}, \frac{1}{3}, \frac{2}{3}, \frac{5}{6}$. Here is one of such formulas:

Proposition.

$$\log G_2\left(\frac{1}{3}\right) = \frac{1}{9} + \frac{\log 3}{72} + \frac{\pi}{18\sqrt{3}} - \frac{2}{3} \log \Gamma\left(\frac{1}{3}\right) - \frac{4}{3} \log A - \frac{\psi^{(1)}\left(\frac{1}{3}\right)}{12\sqrt{3}\pi} \quad (12)$$

The Barnes formula (11) had been generalized by Vardi to $G_n\left(\frac{1}{2}\right)$. The formula is too complicated to be demonstrated here, I will show only one particular case, when $n = 3$:

$$\log G_3\left(\frac{1}{2}\right) = \frac{1}{8} + \frac{\log 2}{24} - \frac{3 \log \pi}{16} - \frac{3}{2} \log A_1 - \frac{7}{8} \log A_2 \quad (13)$$

What is unknown is if there are any other special cases for the triple Barnes function. However, if that is so (though I doubt it, the reason lies in antisymmetry of integral representations (8) and (9)), then all of them should follow from

$$\begin{aligned} \int_0^z \log G_2(x) dx = \\ 2 \log G_3(z+1) + z \log G_2(z) + \frac{z(z-2)}{4} \log(2\pi) - 2z \log A - \frac{z(2z^2 - 6z + 3)}{12} \end{aligned} \quad (14)$$

Remark. Proposition (12) can be generalized to all even orders of the G function.

Remark. We can encounter a similar symmetry mismatch in Bernoulli and Euler polynomials, polygamma and Zeta functions.

Relation to the Hurwitz function

If you feel there are too many special functions in my talk, then I refer you to Berry's recent article at

<http://physicstoday.org/pt/vol-54/iss-4/p11.html>

Berry favors the special function; he wrote, "The persistence of special functions is puzzling as well as surprising." People with the internet connection can read that article now, but the rest of us ...

In the previous section we showed that

$$\sum_{m=1}^n m \log m = n \log n! - \log G_2(n+1) \quad (15)$$

On the other hand, by using an analytic property of the Hurwitz zeta function, the sum in the left hand can be expressed as

$$\sum_{k=1}^n k \log k = \lim_{s \rightarrow -1} \frac{d}{ds} \left(\sum_{k=1}^{\infty} \frac{1}{k^s} - \sum_{k=n+1}^{\infty} \frac{1}{k^s} \right) = \zeta'(-1, n+1) - \zeta'(-1) \quad (16)$$

where $\zeta'(t, z) = \frac{d}{dt} \zeta(t, z)$. Now compare (15) with (16), we obtain

$$\log G_2(z+1) - z \log \Gamma(z) = \zeta'(-1) - \zeta'(-1, z), \quad \text{Re}(z) > 0 \quad (17)$$

In a similar way, we derive the relationship between the triple Barnes function and the Hurwitz function:

$$2 \log G_3(z) = (z^2 - 3z + 2)(\zeta'(0) - \zeta'(0, z)) + (3 - 2z)(\zeta'(-1) - \zeta'(-1, z)) + \zeta'(-2) - \zeta'(-2, z) \quad (18)$$

The formula suggests the pattern:

$$\log G_n(z) = \frac{1}{(n-1)!} \sum_{k=0}^{n-1} P_{k,n}(z) (\zeta'(-k) - \zeta'(-k, z)) \quad (19)$$

with the question what are the polynomials $P_{k,n}(z)$? In 1988, Vardi derived an implicit form for them (he used macsyma!). However, as it turns out these polynomials have quite a nice explicit form. As usual, I'll skip the proof (well, the proof is in the fine print at the end of my talk)

$$P_{k,n}(z) = \sum_{i=k+1}^n (-z)^{i-k-1} \binom{i-1}{k} \left[\begin{matrix} n \\ i \end{matrix} \right] \quad (20)$$

where $\left[\begin{matrix} n \\ i \end{matrix} \right]$ are unsigned Stirling numbers of the first kind (I am using a notation proposed by Knuth). The polynomials $P_{k,n}(z)$ satisfy the functional equation

$$P_{k,n+1}(z) - n P_{k,n}(z) - P_{k,n+1}(z+1) = 0$$

$$P_{k,n}(z) = 0, \quad k \geq n$$

Here are the main properties of these polynomials:

$$\begin{aligned}
P_{n-1,n}(z) &= 1 & P_{k,n}(1) &= \begin{bmatrix} n-1 \\ k \end{bmatrix} \\
P_{n-2,n}(z) &= z(1-n) + \frac{n(n-1)}{2} & P_{0,n}(z) &= \begin{pmatrix} n-z-1 \\ n-1 \end{pmatrix} (n-1)!
\end{aligned} \tag{21}$$

The formula (19) has an immediate consequence for $z = \frac{1}{2}$:

Proposition.

$$\begin{aligned}
(n-1)! \log G_n\left(\frac{1}{2}\right) &= -\frac{(2n-3)!! \log \pi}{2^n} + \log 2 \sum_{k=1}^n \frac{P_{k,n}\left(\frac{1}{2}\right) B_{k+1}}{(k+1) 2^k} + \\
&\quad \sum_{k=1}^n P_{k,n}\left(\frac{1}{2}\right) \frac{2^{k+1} - 1}{2^k} \zeta'(-k)
\end{aligned} \tag{22}$$

It is worth noting that $P_{k,n}\left(\frac{1}{2}\right)$ can be directly computed as coefficients by x^k in expansion of

$$2^{n-k-1} \prod_{j=1}^{n-1} (x + 2j - 1)$$

Remark: Representation (22) is much simpler than one obtained by Vardi.

Regarding (19), it is natural to ask, what would be the inverse:

$$\zeta'(-n) - \zeta'(-n, z) = \frac{1}{(n-1)!} \sum_{k=0}^{n-1} R_{k,n}(z) \log G_k(z) \tag{23}$$

For example, for $n = 2$ we have

$$\zeta'(-2) - \zeta'(-2, z) = -2 \log G_3(z+1) + (1-2z) \log G_2(z+1) + z^2 \log \Gamma(z)$$

I don't have a closed form representation for polynomials $R_{k,n}(z)$.

This formula has a computational interest. Since the computational complexity of the Barnes function is less than the derivatives of the Hurwitz function, the formula (23) can be used for fast numeric computation of the derivatives of the Hurwitz function. Another interest of (23) is that it will give an analytic continuation of the Hurwitz function to the whole complex z -plane.

Numerical Strategies

We have presented here many available results on the Barnes function as well as new results on numeric and symbolic computations. In this section I will discuss a numeric computational scheme for the Barnes G function. The integral representation via polygamma functions (see my paper at ISSAC'01) seems to be an efficient numeric procedure for evaluating the G function.

$$\log G_2(z+1) = \frac{z-z^2}{2} + z \log(\sqrt{2\pi}) + \int_0^\infty x \psi(x) dx, \quad R(z) > -1 \tag{24}$$

This representation demonstrates that the complexity of computing $G_2(z)$ depends at most on the complexity of computing the polygamma function. The restriction $R(z) > -1$ can be easily removed by analyticity of the polygamma. For example, by resolving the singularity of the integrand at the pole $x = -1$, we continue $\log G_2(z+1)$ to the wider area $R(z) > -2$:

$$\log G_2(z+1) = \frac{z-z^2}{2} + z \log(\sqrt{2\pi}) + \log(z+1) + \int_0^z x \psi(x) - \frac{1}{x+1} dx \quad (25)$$

Another way to continue $G_2(z+1)$ into the left half-plane is to use the identity

$$\psi(x) = \psi(-x) - \pi \cot(\pi x) - \frac{1}{x}$$

which upon substituting it into (22) yields

$$\log G_2(1+z) = \log G_2(1-z) + z \log(2\pi) - \int_0^z \pi x \cot(\pi x) dx \quad (26)$$

This identity holds everywhere in a complex plane of z , except the real axes, where the integrand has simple poles. Therefore, in view of (24) we can continue $G_2(z+1)$ to $\mathbb{C} / \mathbb{R}^-$. Note, if z is negative we can still use the representation (22), but with a contour of integration deformed in such a way that it does not cross the poles.

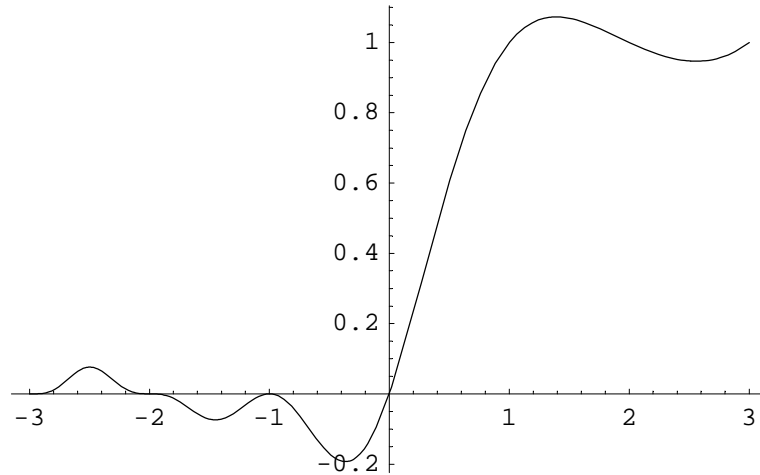
Based on the aforementioned statement, we define the double Barnes function $G_2(z)$ as

$$G_2(z) = (2\pi)^{\frac{z-1}{2}} \exp\left(-\frac{(z-1)(z-2)}{2} + \int_0^{z-1} x \psi(x) dx\right), \quad \arg(z) \neq \pi \quad (27)$$

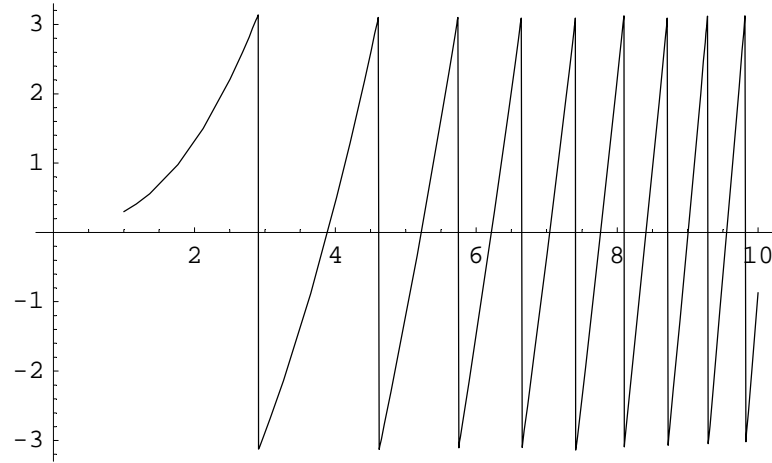
This representation is valid for $z \in \mathbb{C} / \mathbb{R}^-$. If z is a negative real, we have

$$G_2(-z) = (-1)^{\lfloor z/2 \rfloor - 1} G_2(z+2) \left(\frac{|\sin(\pi z)|}{\pi} \right)^{z+1} \exp\left(\frac{1}{2\pi} \text{Cl}_2(2\pi(z - \lfloor z \rfloor)) \right) \quad (28)$$

where $\lfloor z \rfloor$ is the floor function, and $\text{Cl}_2(z)$ is the Clausen function. Here is a picture of the G function over the interval $(-3,3)$:



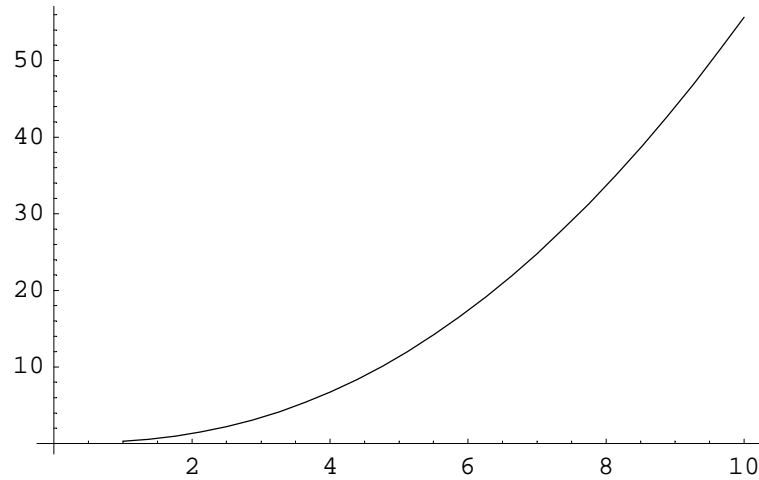
There are many applications, particularly in number theory, where the logarithm of the Barnes function often appears. However, because of a branch cut of logarithm, the function $\log G_2(z)$ includes spurious discontinuities for complex argument. The following is a picture of $\text{Im}(\log G_2(2 - i y))$, where $y \in \{1, 10\}$ demonstrates this:



Therefore, we define an additional function $\text{LOG_G}(z)$ (the logarithm of the Barnes function) which is an analytic function throughout the complex z plane

$$\text{LOG_G}(z) = -\frac{(z-1)(z-2)}{2} + \frac{z-1}{2} \log(2\pi) + \int_0^{z-1} x \psi(x) dx \quad (29)$$

Here is a picture of $\text{Im}(\text{LOG_G}(2 - i y))$, where $y \in \{1, 10\}$:



Functional Determinants

Consider the n -dimensional sphere S^n and the Laplacian operator Δ_n on a manifold. The determinant of the Laplacian is defined as the product of nonzero eigenvalues of the Laplacian:

$$\det \Delta_n = \prod_{k=1}^{\infty} \lambda_k$$

where $\lambda_1, \lambda_2, \dots$ are eigenvalues. In order to regularize this divergent product, we introduce an auxillary function

$$F_n(s) = \sum_{k=1}^{\infty} \frac{1}{\lambda_k^s}$$

Differentiating both sides of it with respect to s and then exponentiating them

$$e^{F'_n(s)} = \prod_{k=1}^{\infty} \exp\left(\frac{-\log \lambda_k}{\lambda_k^s}\right)$$

Setting $s=0$, we formally obtain

$$e^{F'_n(0)} = \prod_{k=1}^{\infty} \exp(-\log \lambda_k) = \prod_{k=1}^{\infty} \frac{1}{\lambda_k}$$

and thus

$$\det \Delta_n = e^{-F'_n(0)}$$

where the divergent sum

$$F'_n(0) = \sum_{k=1}^{\infty} \log \lambda_k$$

is understood as analytic continuation, and defined by the process of zeta regularization (see Voros).

The eigenvalues of the Laplacian operator on the sphere are known to be $k(k+n-1)$. Taking into account the multiplicity of eigenvalues, the outcome is

$$F_n(s) = \sum_{k=1}^{\infty} \frac{\binom{k+n}{n} - \binom{k+n-2}{n}}{k^s (k+n-1)^s} \quad (30)$$

Computation of $\det \Delta_n$ is equivalent to computing derivatives of (14) at $s=0$. We regularize the divergent series $F'_n(0)$ by means of the derivatives of the Hurwitz function. Before proceeding with the general case, I will outline the idea of computing $F'_n(0)$ for $n=2$.

If $n=2$, the formula (29) simplifies to

$$F_2(s) = \sum_{k=1}^{\infty} \frac{2k+1}{k^s (k+1)^s} = \lim_{N \rightarrow \infty} \sum_{k=1}^N \frac{2k+1}{k^s (k+1)^s}$$

Differentiating it with respect to s and letting $s = 0$, we have

$$F'_2(0) = - \lim_{N \rightarrow \infty} \sum_{k=1}^N (2k+1) (\log k + \log(k+1)) \quad (31)$$

or, in alternative form,

$$F'_2(0) = - \lim_{N \rightarrow \infty} \left[(2N+1) \log(N+1) + 4 \sum_{k=1}^N k \log k \right]$$

Using the analytic property of the Hurwitz function,

$$\sum_{k=1}^N k \log k = \zeta'(-1, N+1) - \zeta'(-1)$$

where $\zeta'(t, z) = \frac{d}{dz} \zeta(t, z)$, we finally obtain

$$F'_2(0) = - \lim_{N \rightarrow \infty} \left[(2N+1) \log(N+1) + 4 \zeta'(-1, N+1) - 4 \zeta'(-1) \right] \quad (32)$$

The asymptotic expansion of $\zeta(s, N)$, when $N \rightarrow \infty$, is well-known (see W.Magnus, F. Oberhettinger, R.P. Soni):

$$\zeta(s, N) = \frac{N^{1-s}}{s-1} + \frac{N^{-s}}{2} + \sum_{j=1}^{m-1} \frac{B_{2j} \Gamma(s+2j-1)}{(2j)! \Gamma(s)} N^{-1-j-s+1} + O\left(\frac{1}{N^{2m+s+1}}\right)$$

Differentiating this with respect to s and setting s to -1 , we get an asymptotic for the derivative:

$$\zeta'(-1, N) = \frac{N^2 \log N}{2} - \frac{N \log N}{2} + \frac{\log N}{12} - \frac{N^2}{4} + \frac{1}{12} + O\left(\frac{1}{N^2}\right) \quad (33)$$

Substituting this into (31), immediately gives

$$F'_2(0) = -4 \left(\frac{1}{12} - \zeta'(-1) \right) = -4 \log A = -0.995017... \quad (34)$$

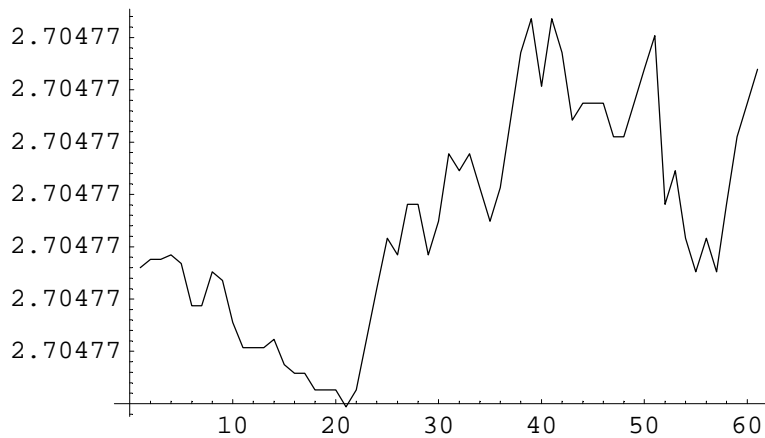
where A is Glaisher's constant. Thus, the determinant of the Laplacian for $n = 2$ is

$$\det \Delta_2 = \exp(-F'_2(0)) = A^4 = \exp\left(\frac{1}{3} - 4 \zeta'(-1)\right) = 2.704772... \quad (35)$$

This formula does not match formulas obtained by Vardi (1988), Quine and Choi (1996), and Kumagai (1999). Their determinant is equal to

$$A^4 e^{\frac{1}{6}} = 3.1953114...$$

Since a regularization is never unique, in the spirit of things, I will run a computer experiment and compute $\det \Delta_2$. We take the formula (30), subtract the asymptotics (32) and plot the \exp of it over the interval $n \in (2000, 5000)$:



Remark. The general formula for $\det \Delta_n$ (I don't have time to show it here) agrees with Vardi and others in odd dimensions, but it's off by a small rational in even dimensions.

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