Envelope computation in dynamic geometry systems

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Abstract

Considerable attention has been given to the computation of geometric loci in dynamic geometry, from both graphical and equational viewpoints. Recent work has established a rigourous approach to the last issue, solving the subject for the algebraic realm. Nevertheless, although the automatic computation of envelopes could be seen as an analogous problem, there are cases where unexpected difficulties emerge.

In this talk we review the state of the art of common dynamic geometry software when dealing with envelopes. Despite their maturity in other subjects, envelopes are generally considered as purely graphic objects in such environments, without any analytic knowledge about them.

We also describe a data structure needed to cope with envelopes in dynamic geometry, and a Sage program able to compute usual envelopes in an efficient way. Finally, in order to show the complexities of such computations, we discuss the envelope of a simple family of ellipses. The search for such envelope will illustrate two facts:

1. There is no general agreement between dynamic geometry developers about the definition of envelope, and
2. Currently, the simple application of computer algebra techniques is not enough to automatically solve the problem.

Keywords

Dynamic Geometry, Automated Deduction in Geometry, Envelope Computation

1 Introduction

Given a family of curves $C_\alpha : F(x, y, \alpha) = 0$, its *envelope* or *discriminant* is defined [1] as the set

$$\mathcal{D} = \{(x, y) \in \mathbb{R}^2 : \text{there exists}\, \alpha \in \mathbb{R} \,\text{with}\, F(x, y, \alpha) = \frac{\partial F}{\partial \alpha} = 0\}.$$

Other definitions coexist with this one, for instance

- The envelope E_1 is the limit of intersections of nearby curves C_α.
- The envelope E_2 is a curve tangent to the C_α.
- The envelope E_3 is the boundary of the region filled by the curves C_α.

where it can be proved that $E_i \subset \mathcal{D}, i = 1, \ldots, 3$. While E_1 seems to be the interpretation of envelopes given by Lagrange to singular solutions of differential equations (see [2]), E_3 and, to a lesser extent, E_2 are behind the intuitive notion of envelope used in dynamic geometry environments. Since these systems are mostly based on a graphical simulation of geometry, their ability to trace geometrc elements is succesfully used to suggest envelopes. Consider, for instance, the family of ellipses with foci in $A(4, 0)$, $B(0, \alpha)$ (a semifree point on the y axis) and major axis with length 5.

If a user activates the trace option for the variable ellipse and moves the point B along its path, a plane region is drawn (Figure 1), the border being the sought envelope, if the third alternative

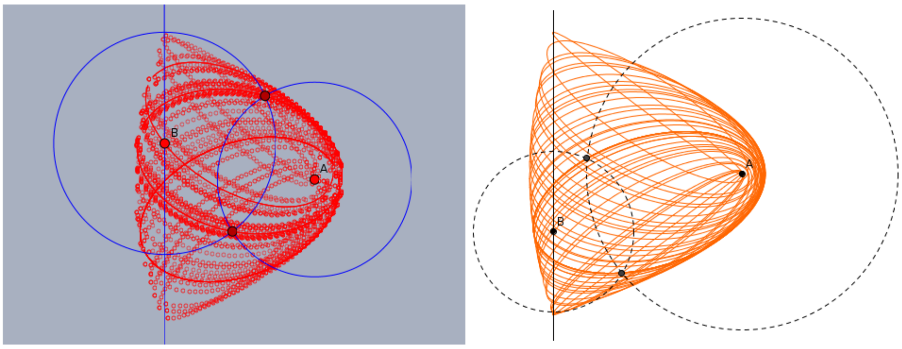

Figure 1: A family of ellipses obtained as loci (left: Cinderella, right: GeoGebra) is traced to suggest its envelope.

definition is used. Nevertheless, as usual in dynamic geometry, the user gets a cursory knowledge: no result about the type of curves defining the border is returned. Even worse, consider a region with holes, or without border. In this last case, the strategy of tracing the family curves could mislead an inexperienced user, as exemplified by searching for the envelope of straight lines passing through the origin.

It should be noted that although there exist a well known dynamic geometry software, Cabri, able to compute equations for constructed objects, its numerical approach is not robust and does not return any result for the above family.

2 A parametric approach for the envelope problem

Several authors (see, for instance, [3, 4]) have proposed extending the data structure of dynamic geometry systems to include what can be called a *parametric description* of constructions. This description involves keeping the relation between primitive and dependent objects in such a way that any assignment of the free variables (i.e., coordinates of basic points and equations of other initial objects) would trigger the *actual* computation of dependent objects.

Using the Sage library in [5], a generic ellipse of the family could be defined by

```
FreePoint('A',4,0)
FreePoint('P1',0,0)
FreePoint('P2',0,1)
Line('y','P1','P2')
PointOnObject('B','y')
FreePoint('M',2,2)
FreePoint('N',2,7)
Line('MN','M','N')
PointOnObject('P','MN')
Circle('c1','A','M','P')
Circle('c2','B','N','P')
IntersectionObjectObject('X','c1','c2')
Locus2('loc','X','B','P')
```

where the `Locus2` function contains a parametric representation of the ellipse. That is, its algebraic description is not a function in two variables, but it also contains the parameter of the variable point B. The polynomial of the family is

$$4y^2\alpha^2 - 4y\alpha^3 - 36x^2 - 100y^2 + \alpha^4 - 32xy\alpha + 16x\alpha^2 + 164y\alpha - 82\alpha^2 + 144x + 81,$$

and one could then use the first definition of envelope to find its equation. The elimination of α returns

$$x^2y^4 + y^6 - 16x^3y^2 - 24xy^4 - 36x^4 + 74x^2y^2 - 2y^4 + 432x^3 + 32xy^2 - 1647x^2 - 207y^2 + 1656x + 1296,$$

and, after factoring,

$$(y^2 - 18x - 9)(y^2 + 2x - 9)(x^2 + y^2 - 8x + 16).$$

Thus, the border of the family of elipses consists of (part) of the above parabolas. Nevertheless,...

3 Things are not so simple

There is a third factor in the expression of the discriminant that is not part of the border. In fact, this factor is the focus A! While understanding why this point appears as part of the discriminant has not been a trivial task (as will be illustrated in the talk), we note that asking, for instance, Wolfram|Alpha, should give a hint about what is happening (Figure 2).

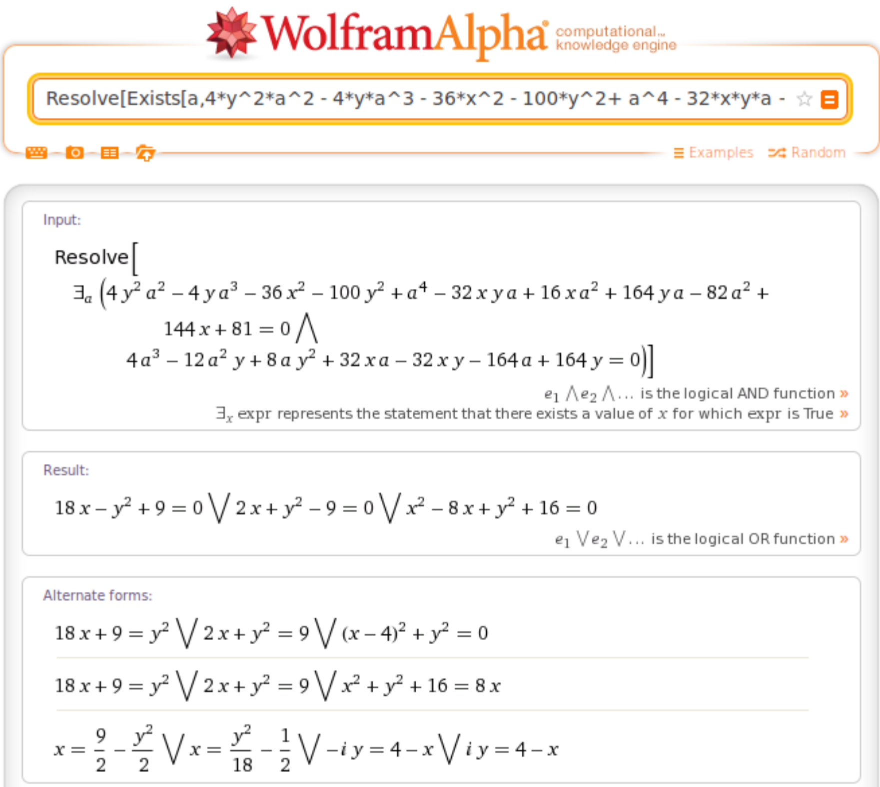

Figure 2: Wolfram|Alpha suggests that point $A(4,0)$ comes from complex components of the envelope.

The moral of this short note is the need of a most rigorous approach when applying algebraic methods valid in $\mathbb{C}$ to misunderstood situations. Here, the family of ellipses is semialgebraic. Thus, automated approaches relying on complex approaches should used with caution!

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