Properties of the Simson–Wallace locus applied on a skew quadrilateral

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The well-known Simson–Wallace theorem reads [3]:

Let $K, L, M$ be orthogonal projections of a point $P$ onto the sides of a triangle $ABC$. Then the locus of $P$ such that $K, L, M$ are collinear, is the circumcircle of $ABC$.

This theorem has several generalizations [4], [5], [10], [6], [7], [9]. A generalization of the Simson–Wallace theorem which is by [2] ascribed to J. D. Gergonne is as follows:

Let $K, L, M$ be orthogonal projections of a point $P$ onto the sides of a triangle $ABC$. Then the locus of $P$ such that the area of the triangle $KLM$ is constant, is the circle through $P$ which is concentric with the circumcircle of $ABC$.

If we consider a tetrahedron $ABCD$ instead of a triangle $ABC$ then we can investigate the locus of points $P \in \mathbb{R}^3$ whose orthogonal projections onto the faces of $ABCD$ are coplanar or form a tetrahedron of a constant volume. This was studied in [10], [6], [7], [9].

The generalization of Simson–Wallace theorem on skew quadrilaterals in the Euclidean 3D space is as follows [6], [8]:

The locus of a point $P$ whose orthogonal projections $K, L, M, N$ onto the sides on a skew quadrilateral $ABCD$ form a tetrahedron of a constant volume $s$ is a cubic surface $G$.


The cubic surface $G$ can be investigated from various points of view. In [8] reducibility of $G$ with $s = 0$ was explored. The following conjecture was stated: The Simson–Wallace locus which is a cubic surface $G$ is decomposable iff two pairs of sides a skew quadrilateral $ABCD$ are of equal lengths. If for instance $|AB| = |BC| = a$ and $|CD| = |DA| = b$, then in the case $a \neq b$ the cubic $G$ decomposes into a plane and a one-sheet hyperboloid, and if $a = b$ we get three mutually orthogonal planes.

In the talk further properties of $G$ are studied. It is well known that the maximum number of lines of a general cubic surface is 27. There is a question how many lines do lie on the cubic $G$? It seems that the maximum number of lines lying on $G$ is 15. This issue is also connected with the number of the so called
tritangent planes which intersect the cubic surface in three lines. Knowing these planes enables us to express $G$ in the form of sum of two cubics which resolve into the product of three linear factors which describe the tritangent planes.

References


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