Properties of the Simson–Wallace locus applied on a skew quadrilateral

P. Pech[1]

[1] *University of South Bohemia, Czech Republic, pech@pf.jcu.cz*

The well-known Simson–Wallace theorem reads [3]:

Let K, L, M be orthogonal projections of a point P onto the sides of a triangle ABC. Then the locus of P such that K, L, M are collinear, is the circumcircle of ABC.

This theorem has several generalizations [4], [5], [10],[6], [7], [9]. A generalization of the Simson–Wallace theorem which is by [2] ascribed to J. D. Gergonne is as follows:

Let K, L, M be orthogonal projections of a point P onto the sides of a triangle ABC. Then the locus of P such that the area of the triangle KLM is constant, is the circle through P which is concentric with the circumcircle of ABC.

If we consider a tetrahedron $ABCD$ instead of a triangle ABC then we can investigate the locus of points $P \in E^3$ whose orthogonal projections onto the faces of $ABCD$ are coplanar or form a tetrahedron of a constant volume. This was studied in [10], [6], [7], [9].

The generalization of Simson–Wallace theorem on *skew quadrilaterals* in the Euclidean 3D space is as follows [6], [8]:

The locus of a point P whose orthogonal projections K, L, M, N onto the sides on a skew quadrilateral $ABCD$ form a tetrahedron of a constant volume s is a cubic surface G.

By searching for the locus and its properties we applied computer aided coordinate method based on Groebner bases computation and Wu–Ritt method using the software CoCoA [1] and the Epsilon library [11] working under Maple.

The cubic surface G can be investigated from various points of view. In [8] reducibility of G with $s = 0$ was explored. The following conjecture was stated: The Simson–Wallace locus which is a cubic surface G is decomposable iff two pairs of sides a skew quadrilateral $ABCD$ are of equal lengths. If for instance $|AB| = |BC| = a$ and $|CD| = |DA| = b$, then in the case $a \neq b$ the cubic G decomposes into a plane and a one-sheet hyperboloid, and if $a = b$ we get three mutually orthogonal planes.

In the talk further properties of G are studied. It is well known that the maximum number of lines of a general cubic surface is 27. There is a question how many lines do lie on the cubic G? It seems that the maximum number of lines lying on G is 15. This issue is also connected with the number of the so called tritangent planes which intersect the cubic surface in three lines. Knowing these planes enables us to express G in the form of sum of two cubics which resolve into the product of three linear factors which describe the tritangent planes.

References

[1] Capani, A., Niesi, G., Robbiano, L.: *CoCoA, a System for Doing Computations in Commutative Algebra*. `http://cocoa.dima.unige.it`

[2] Chou, S. C.: *Mechanical Geometry Theorem Proving*. D. Reidel Publishing Company, Dordrecht (1987).

[3] Coxeter, H. S. M., Greitzer, S. L: *Geometry revisited*, Toronto New York (1967).

[4] Giering, O.: *Affine and Projective Generalization of Wallace Lines*, J. Geometry and Graphics **1**, 119-133 (1997).

[5] Guzmán, M:*An Extension of the Wallace–Simson Theorem: Projecting in Arbitrary Directions*, Amer. Math. Monthly **106**, 574-580 (1999).

[6] Pech, P.: *On Simson–Wallace Theorem and Its Generalizations*, J. Geometry and Graphics **9**, 141-153 (2005).

[7] Pech, P.: *On a 3D extension of the Simson–Wallace theorem*, J. Geometry and Graphics, **18**, 205-215 (2014).

[8] P. Pech: *Extension of Simson–Wallace theorem on skew quadrilaterals and further properties*, in *Lecture Notes in Artificial Intelligence* (ADG-2014), Springer 2015, to appear.

[9] Riesinger, R.: *On Wallace Loci from the Projective Point of View*, J. Geometry and Graphics **8**, 201-213 (2004).

[10] Roanes–Lozano, E., Roanes–Macías, E.: *Automatic Determination of Geometric Loci. 3D-Extension of Simson–Steiner Theorem*, in *Lecture Notes in Artificial Intelligence* (AISC 2000), **1930**, pp. 157-173.

[11] Wang, D.: *Epsilon: A library of software tools for polynomial elimination*, in: *Mathematical Software*, (Cohen, A., Gao, X. S., Takayama, N., eds), pp. 379–389. World Scientific, Singapore New Jersey (2002). `http://www-calfor.lip6.fr/~wang/epsilon/`