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What is the integral of x^n ?

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Thus, computer algebra systems usually solve a problem under the implied assumption that any parameters appearing in the problem will take values that lead to the general result returned. The title refers to the fact that the systems return the integral of x^n as $x^{n+1}/(n+1)$ while omitting the condition $n \neq -1$.

We shall use the word *specialization* to describe the action of substituting specific values (usually numerical, but not necessarily) into a formula. The *specialization problem* is a label for a cluster of problems associated with formulae and their specializations, the problems ranging from inelegant results to invalid ones. For example, in [2] an example is given in which the evaluation of an integral by specializing a general formula misses a particular case for which a more elegant expression is possible. The focus here, however, is on situations in which specialization leads to invalid or incorrect results. To illustrate the problems, consider

$$I_1 = \int \left(\alpha^{\sigma z} - \alpha^{\lambda z}\right)^2 \, dz = \frac{1}{2\ln\alpha}\left(\frac{\alpha^{2\lambda z}}{\lambda} + \frac{\alpha^{2\sigma z}}{\sigma} - \frac{4\alpha^{(\lambda+\sigma)z}}{\lambda+\sigma}\right) . \tag{1}$$

Expressions equivalent to this are returned by Maple, Mathematica and many other systems, such as the Matlab symbolic toolbox. It is easy to see that the specialization $\sigma = 0$ leaves the integrand in (1) well defined, but the expression for its integral on the right-hand side is no longer defined. If we pursue this further, we see that there are multiple specializations for which (1) fails, *viz.* $\alpha = 0$, $\alpha = 1$, $\lambda = 0$, $\sigma = 0$, $\lambda = -\sigma$, and combinations of these. The question of how or whether to inform computer users of these special cases has been discussed in the CAS literature many times [1]. A list of every special case for (1) is as follows.

$$I_1 = \begin{cases} \dfrac{1}{2\lambda\ln\alpha}\left(\alpha^{2\lambda z} - \alpha^{-2\lambda z} - 4z\lambda\ln\alpha\right) , & \left[\begin{array}{l}\lambda+\sigma = 0\,,\\ \alpha\neq 0\,, \alpha\neq 1\,, \sigma\neq 0\,;\end{array}\right. \\ z + \dfrac{1}{2\lambda\ln\alpha}\left(\alpha^{\lambda z}(\alpha^{\lambda z} - 4)\right) , & \left[\begin{array}{l}\sigma = 0\,,\\ \alpha\neq 0\,, \alpha\neq 1\,, \lambda\neq 0\,;\end{array}\right. \\ z + \dfrac{1}{2\sigma\ln\alpha}\left(\alpha^{\sigma z}(\alpha^{\sigma z} - 4)\right) , & \left[\begin{array}{l}\lambda = 0\,,\\ \alpha\neq 0\,, \alpha\neq 1\,, \sigma\neq 0\,;\end{array}\right. \\ \text{ComplexInfinity} , & \left[\begin{array}{l}\alpha = 0\,,\\ \Re(\lambda z)\,\Re(\sigma z) < 0\,;\end{array}\right. \\ \text{Indeterminate} , & \left[\begin{array}{l}\alpha = 0\,,\\ \Re(\sigma z)\,\Re(\lambda z) \geq 0\,;\end{array}\right. \\ \dfrac{1}{2\ln\alpha}\left(\dfrac{\alpha^{2\lambda z}}{\lambda} + \dfrac{\alpha^{2\sigma z}}{\sigma} - \dfrac{4\alpha^{(\lambda+\sigma)z}}{\lambda+\sigma}\right) , & \text{otherwise, (generic case)} . \end{cases} \tag{2}$$

Expressions such as this will be called *comprehensive antiderivatives*. There are several questions surrounding such expressions. The first is whether comprehensive antiderivatives should be returned to users. A second question is how systems can compute such expressions. The automatic discovery of exceptional cases is not easy. A third question concerns *continuity with respect to parameters.*

We shall discuss why the expression

$$\int x^n \, dx = \frac{x^{n+1}}{n+1} - \frac{1}{n+1}$$

is better than the usual expression, and how we found it.

Keywords: Specialization problem, integration, parameters, continuity

References

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