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Familiarizing students with definition of Lebesgue measure using Mathematica - some examples of calculation directly from its definition

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"Young man, in mathematics you don't understand things. You just get used to them"

John von Neumann

In this talk we present some examples of calculation the Lebesgue measure of some subsets of $\mathbb{R}^2$ directly from definition. We cannot find such examples in the literature we know. We will consider the following subsets of $\mathbb{R}^2$: $\{(x,y) \in \mathbb{R}^2 : 0 \le y \le x^2, 0 \le x \le 1\}$, $\{(x,y) \in \mathbb{R}^2 : 0 \le y \le \sin x, 0 \le x \le \pi/2\}$, $\{(x,y) \in \mathbb{R}^2 : 0 \le y \le \exp(x), 0 \le x \le 1\}$, $\{(x,y) \in \mathbb{R}^2 : 0 \le y \le \ln(1-2r\cos x + r^2), 0 \le x \le \pi\}$, $r > 1$. The aim of these examples is to familiarize students with the definition of Lebesgue measure. We calculate sums, limits and plot graphs and dynamic plots of needed sets and unions of rectangles sums of which volumes approximate Lebesgue measure of the sets, using Mathematica. The title of this talk is very similar to the title of author's article [1] which deals with definition of Lebesgue integral but our talk deals with definition of Lebesgue measure instead. Using Mathematica or others CAS programs for calculation Lebesgue measure directly from its definitions, seems to be didactically useful for students because of the possibility of symbolic calculation of sums, limits - checking our hand calculations and plot dynamic graphs. Moreover we get students used not only to definition of Lebesgue measure but also to CAS applications generally.

The following definitions we will use in our talk (see [9, 3]):

Rectangles. A closed rectangle R in $\mathbb{R}^d$ is given by the product of d one-dimensional closed and bounded intervals: $R = [a_1,b_1] \times [a_2,b_2] \times \cdots \times [a_d,b_d]$, where $a_j \le b_j$ are real numbers, $j = 1,2,\ldots,d$. In other words, we have $R = \{(x_1,\ldots,x_d) \in \mathbb{R}^d : a_j \le x_j \le b_j$ for all $j = 1,2,\ldots,d\}$. We remark that in our definition, a rectangle is closed and has sides parallel to the coordinate axis. In $\mathbb{R}$, the rectangles are precisely the closed and bounded intervals, while in $\mathbb{R}^2$ they are the usual four-sided rectangles. In $\mathbb{R}^3$ they are the closed parallelepipeds.

We say that the lengths of the sides of the rectangle R are $b_1 - a_1, \ldots, b_d - a_d$. The volume of the rectangle R is denoted by $\operatorname{vol}(R)$, and is defined to be $\operatorname{vol}(R) = (b_1 - a_1)\cdots(b_d - a_d)$.

Definition 1. (see [3, 7, 8, 9]) Let $(\mathbb{R}^2, \mathfrak{M}, m)$ be measure space, where $\mathfrak{M}$ is $\sigma-$ algebra of Lebesgue measurable subsets in $\mathbb{R}^2$, and m- Lebesgue measure on $\mathbb{R}^2$. The measure m for any $A \in \mathfrak{M}$ is defined by the following formula:

$$m(A) = \inf\Big\{ \sum_{j=1}^{\infty} \operatorname{vol}(R_j) : A \subset \bigcup_{j=1}^{\infty} R_j,\ \ R_j \text{ is closed rectangle in } \mathbb{R}^2, j \in \mathbb{N} \Big\}. \quad (1)$$

Keywords: Higher education, Lebesgue measure, Application of CAS, Mathematica, Mathematical didactics

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