Accurate plotting in 3D: how to choose the mesh

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When studying continuity of 1-variable functions, a classical example is given by the function f such that $f(x) = \sin \frac{1}{x}$. this is an opportunity to show an example of Heine's Theorem:

Theorem: *Let f be a function defined on a punctured neighborhood of the real number a. The two following properties are equivalent:*

1. *The function f has a limit l at a (maybe infinite).*

2. *For every convergent sequence (x_n) of real numbers whose limit is a, the sequence $(f(x_n))$ has limit l.*

Take the sequences given by

$$x_n = \frac{1}{2n\pi} \text{ and } y_n = \frac{1}{2n\pi + \frac{\pi}{2}}$$

Both sequences converge and have a limit equal to 0, but $f(x_n)$ and $f(y_n)$ have different limits, showing that f has a discontinuity at 0.

This situation cannot be illustrated using a plot. Figure 1 shows the strange plot obtained using a standard command*. Zooming does not improve the visualization, as the function has an infinite number of oscillations within a compact interval around 0. Students have difficulties both with the visualization and with "for every sequence". The teacher has to insist on the fact that this theorem is mostly used to disprove continuity/existence of a limit.

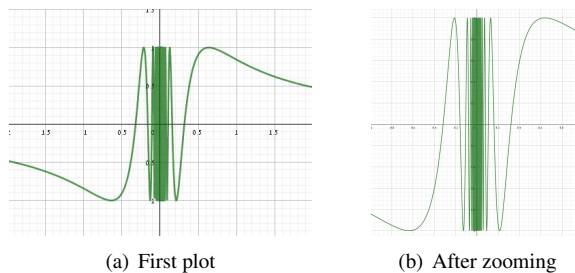(a) First plot   (b) After zooming

Figure 1: Strange plots for a one-variable function

Transition towards Calculus II, with functions of 2 real variables (or more than two) leads to problems, either different or more of the same. Sometimes, the visualization becomes harder to understand. For example, consider the function given by

$$f(x, y) = \frac{1}{1 - (x^2 + y^2)}. \tag{1}$$
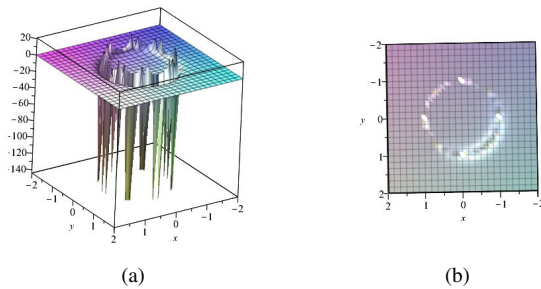
(a)   (b)

Figure 2: Strange plots for a two-variable function

Using standard commands for plotting graphs of 2-variable functions may provide a non accurate result (see Figure 2).

The function has non-isolated singularities, but the plot in cartesian coordinates does not show them. Moreover the plot shows a lot of needles. The reason is that the standard **plot3d** command divided the given domain using a cartesian mesh, made of squares; this can be visualized when looking on the graph "from above", as shown in Figure 2b. Actually the plot is obtained using numerical computations: the CAS computes values of the function on the edges of the cells and make interpolations for what happens inside the cells, as explained in [2]. Zooming is useless, as this only inflates the cells, but does not compute new values for the function inside each cell.

Based on a prior mathematical analysis of the continuity of the function, a more accurate plot may be achieved using a new coordinate systems (See the reference [4]).

We choose new coordinates, which fit the specific coordinates of the given function. Figure 3a shows a plot of the function defined in Equation (1), using polar coordinates, with $x = r\cos\theta$, $y = r\sin\theta$. Even the choice of these coordinates do not ensure that the plot will be really accurate. Figure 3b and Figure 3c represent the same plot, viewed from different directions, after a slight modification of the domain. Here the interpolations once again hide the actual discontinuities.
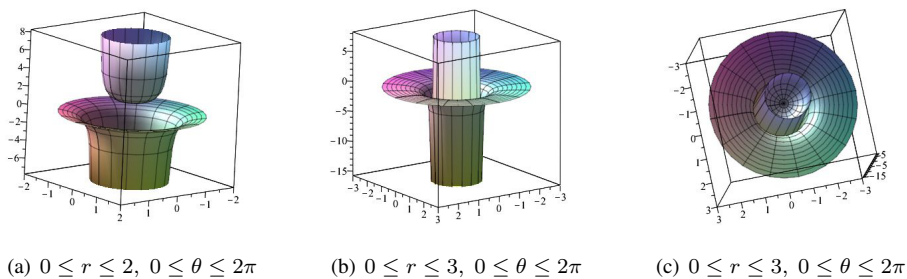

(a) $0 \le r \le 2,\ 0 \le \theta \le 2\pi$ (b) $0 \le r \le 3,\ 0 \le \theta \le 2\pi$ (c) $0 \le r \le 3,\ 0 \le \theta \le 2\pi$

Figure 3: The influence of the choice of the plotting domain

Other modifications, such as considering $g(x, y) = 1/(3 - (x^2 + y^2))$, or $h(x, y) = 1/(1 - (x^2 + 3y^2))$ lead students to understand the need for a mathematical analysis of the data prior to computerized work.

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*We used GeoGebra, but the same phenomenon appears with any other package.

Algorithms have been described in [3] in order for the software to determine a mesh which will provide an accurate plot. In our talk we will present both further advances in this field, and actual situations encountered in classroom. Moreover, we can mention the possibility to enhance more understanding by using Virtual Reality, as described in [1].

Keywords: surface, accurate plot, mesh

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