Math 511
Assignment 3, due Friday, March 9

1. (Wade 12.1.4b) Prove that given $r > 0$, $B_r(0) := \{x \in \mathbb{R}^n : |x| < r\}$ is a Jordan Region.
   Note: One way to start is by observing that it suffices to show that
   $$\{x \in \mathbb{R}^n : x_1 \geq |x|/\sqrt{n} \text{ and } |x| = r\}$$
   has volume zero (though you need to explain why this is sufficient). Then realize this set as the image of
   $$E = \{y \in \mathbb{R}^n : y_1 = 0,\ y_2^2 + \cdots + y_n^2 \leq \left(1 - \frac{1}{n}\right) r^2\}$$
   under the map $F(y_1, \ldots, y_n) = (y_1 + \sqrt{r^2 - (y_2^2 + \cdots + y_n^2)}, y_2, \ldots, y_n)$.

2. (Wade 12.1.5) Let E be a Jordan region in $\mathbb{R}^n$.
   (a) Prove that E^o and $\overline{E}$ are Jordan regions (Recall Rudin, Ch.2, 9(e,f)).
   (b) Prove that $\text{Vol}(E^o) = \text{Vol}(\overline{E}) = \text{Vol}(E)$.
   (c) Prove that $\text{Vol}(E) > 0$ if and only if $E^o \neq \varnothing$.

3. (Wade 12.1.6 (a,b,c)) Suppose that E_1, E_2 are Jordan regions in $\mathbb{R}^n$.
   (a) Prove that if $E_1 \subset E_2$, then $\text{Vol}(E_1) \leq \text{Vol}(E_2)$.
   (b) Prove that $E_1 \cap E_2$ and $E_1 \setminus E_2$ are Jordan regions.
   (c) Prove that if E_1, E_2 are nonoverlapping, then $\text{Vol}(E_1 \cup E_2) = \text{Vol}(E_1) + \text{Vol}(E_2)$.

   On your own, observe that $\text{Vol}(E_1 \setminus E_2) = \text{Vol}(E_1) - \text{Vol}(E_2)$ when $E_1 \subset E_2$ and that more generally, $\text{Vol}(E_1 \cup E_2) = \text{Vol}(E_1) + \text{Vol}(E_2) - \text{Vol}(E_1 \cap E_2)$.

4. (Wade 12.2.9) Suppose that V is open in $\mathbb{R}^n$ and that $f : V \to \mathbb{R}$ is continuous. Prove that if
   $$\int_E f\, dV = 0$$
   for all nonempty Jordan regions $E \subset V$, then $f = 0$ on V.

On your own:

1. (Wade 12.2.3) Let E be an open Jordan region in $\mathbb{R}^n$ and $x_0 \in E$. If $f : E \to \mathbb{R}$ is integrable on E and continuous at x_0, prove that
   $$\lim_{r \to 0^+} \frac{1}{\text{Vol}(B_r(x_0))} \int_{B_r(x_0)} f\, dV = f(x_0).$$

2. (Wade 12.2.5) If $E_0 \subset E$ are Jordan regions in $\mathbb{R}^n$ and $f : E \to \mathbb{R}$ is integrable on E, prove that f is integrable on E_0.