# Math 510, Fall 2017
# Assignment 9, due Wednesday, November 8

Exercises to hand in. As usual, hand in Parts I and II separately:

Part I:

1. Rudin, Chapter 6, #3.
2. Let $f : [a, b] \to \mathbb{R}$ be bounded on $[a, b]$ and Riemann integrable on $[c, b]$ for every $c \in (a, b)$. Prove that $f$ is integrable on $[a, b]$ and that

$$\int_a^b f(x)\,dx = \lim_{c \to a+} \int_c^b f(x)\,dx.$$

Part II:

1. In this problem, consider a real valued function $f : [a, \infty) \to \mathbb{R}$ which is Riemann integrable over any compact subinterval $[c, d] \subset [a, \infty)$. $f$ is said to be *improperly integrable* on $[a, \infty)$ if the limit $\lim_{b\to\infty} \int_a^b f(x)\,dx$ exists as a real number, in which case we denote this limit as $\int_a^\infty f(x)\,dx$.
   (a) Suppose $f$ is a nonnegative function which is improperly integrable on $[a, \infty)$. Show that

   $$\int_a^\infty f(x)\,dx = \sup\left\{\int_a^R f(x)\,dx : R > a\right\}.$$

   (b) Prove that if $|f|$ is improperly integrable on $[a, \infty)$, then so is $f$.
   (c) Prove the *integral test*: Suppose $f : [1, \infty) \to [0, \infty)$ is a nonnegative decreasing function. The series $\sum_{n=1}^\infty f(n)$ converges if and only if $f$ is improperly integrable on $[1, \infty)$.
2. Suppose $f : [a, b] \to \mathbb{R}$ is Riemann integrable. Using Theorem 6.6, show that $f^2$ is also a Riemann integrable function. In other words, show directly that for any $\varepsilon > 0$ there exists a partition $P$ of $[a, b]$ such that

   $$U(P, f^2) - L(P, f^2) < \varepsilon.$$

   You may not apply Theorems 6.11 or 6.13 in the book to this problem.

   Hint: For any bounded function $g$ on $[a, b]$, Exercise 3 in Part II of Assignment 1 shows that taking $A = g([a, b])$ (the image of $[a, b]$ under $g$), we have

   $$\sup_{x\in[a,b]} g(x) - \inf_{x\in[a,b]} g(x) = \sup\{|g(x) - g(y)| : x, y \in [a, b]\}$$
   $$= \sup\{g(x) - g(y) : x, y \in [a, b]\}.$$

On your own: Rudin, Chapter 6: 1, 2, 4, 5, 8 and the following problems:

1. Use Exercise 2 from Part I to show that any function $f : [0, 1] \to \mathbb{R}$ agreeing with $\sin(1/x)$ for $x \neq 0$ is integrable.
2. Find a nonnegative function $f$ which is improperly integrable on $[0, \infty)$, but $f$ is unbounded on that domain.

Reading: Rudin, Chapter 6.