Math 581, Spring 2017
Assignment 7, due Wednesday, April 26

In all problems below, $\mathcal{X}$ is a complex Hilbert space. There is a problem on the back page.

1. Let a be a sesquilinear form on $\mathcal{X}$ satisfying $|a(x,y)| \le C_1\|x\|\,\|y\|$ and $|a(x,x)| \ge C_2\|x\|^2$ for some $C_1, C_2 > 0$ and all $x, y \in \mathcal{X}$. Show there exists a bounded invertible operator B on all of $\mathcal{X}$ such that $\langle x, y\rangle = a(x, By)$ for all x, y.

2. Suppose P_1, P_2 are orthogonal projections onto closed subspaces $\mathcal{M}_1, \mathcal{M}_2 \subset \mathcal{X}$ respectively.

   (a) Prove that $P_2 - P_1$ is an orthogonal projection if and only if $P_1 \le P_2$. Show that in this case, $P_2 - P_1$ is projection onto $\mathcal{M}_1^\perp \cap \mathcal{M}_2$.

   (b) Prove that $P_2P_1 = 0$ if and only if $\mathcal{M}_1 \perp \mathcal{M}_2$.

3. Let $\{\alpha(j)\}_{j=1}^\infty$ be a bounded sequence in $\mathbb{R}$ and define $A : \ell^2 \to \ell^2$ by

$$(Ax)(j) = \alpha(j)x(j).$$

   It is not hard to check that A is bounded, linear, and self-adjoint (do this on your own).

   (a) Show that

$$\inf_{\|x\|=1} \langle Ax, x\rangle = \inf_j \alpha(j), \qquad \sup_{\|x\|=1} \langle Ax, x\rangle = \sup_j \alpha(j).$$

   (b) Find explicit formulas for $|A|$, A^+, and A^-. Be rigorous and appeal to the definitions of these operators given in class.

   (c) Show that if $\{E(\lambda)\}_{\lambda\in\mathbb{R}}$ is the resolution of the identity corresponding to A, then

$$(E(\lambda)x)(j) = \begin{cases} x(j), & \text{if } \alpha(j) \le \lambda \\ 0, & \text{otherwise.} \end{cases}$$

4. Recall that a $\mathbb{C}$-valued function g is of bounded variation (BV) on an interval $[a,b] \subset \mathbb{R}$ if

$$\sup \left\{ \sum_{j=1}^{n} |g(\lambda_j) - g(\lambda_{j-1})| : \{\lambda_0, \lambda_1, \dots, \lambda_n\} \text{ is a partition of } [a,b] \right\} < \infty.$$

It is not hard to check that the functions of bounded variation on $[a,b]$ is a vector space and if any BV function is extended to be constant outside of $[a,b]$ the extension is also BV on any compact interval containing $[a,b]$ (do this on your own). Let $\{E(\lambda)\}_{\lambda \in \mathbb{R}}$ be the resolution of the identity associated to a self-adjoint operator $A \in B(X)$ and set

$$m := \inf_{\|x\|=1} \langle Ax, x \rangle, \qquad M := \sup_{\|x\|=1} \langle Ax, x \rangle.$$

(a) Show that any increasing function on a compact interval is of bounded variation and that for any $x \in \mathcal{X}$, $g(\lambda) := \langle E(\lambda)x, x \rangle$ is increasing on $\mathbb{R}$.

(b) Now fix $x, y \in \mathcal{X}$ and consider $g(\lambda) = \langle E(\lambda)x, y \rangle$. Show that there exists increasing functions $g_i : \mathbb{R} \to \mathbb{R}$, $i = 1, 2, 3, 4$, such that

$$g(\lambda) = g_1(\lambda) - g_2(\lambda) + i(g_3(\lambda) - g_4(\lambda)).$$

Conclude g is of bounded variation on any compact interval in $\mathbb{R}$.

(c) Suppose $a < m < M < b$ and $f : [a,b] \to \mathbb{C}$ is continuous. Since the g_i in the previous part are all increasing, the Riemann-Stieltjes integral $\int_a^b f(\lambda) dg_i$ exists for each $i = 1, 2, 3, 4$ and we may define

$$\int_a^b f(\lambda) dg := \int_a^b f(\lambda) dg_1 - \int_a^b f(\lambda) dg_2 + i \left(\int_a^b f(\lambda) dg_3 - \int_a^b f(\lambda) dg_4 \right)$$

With this interpretation, show that $\langle f(A)x, y \rangle = \int_a^b f(\lambda) dg$.

On your own: verify all the "on your own" elements of the above problems.