ON THE TOPOLOGY OF SOME SASAKI-EINSTEIN MANIFOLDS

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Abstract. This is a sequel to our paper [BTF15] in which we concentrate on developing some of the topological properties of Sasaki-Einstein manifolds. In particular, we explicitly compute the cohomology rings for several cases not treated in [BTF15] and give formulae for homotopy equivalence as well as homeomorphism equivalence in one particular 7-dimensional case.

1. Introduction

Recently the authors have been able to obtain many new results on extremal Sasakian geometry [BTF13c, BTF13a, BTF14a, BTF15] by giving a geometric construction that combines the ‘join construction’ of [BG00, BGO07] with the ‘admissible construction of Hamiltonian 2-forms’ for extremal Kähler metrics described in [ACG06, ACGTF04, ACGTF08b, ACGTF08a]. The current paper is a result of re-arranging the two previous ArXiv papers [BTF13b, BTF14b]. The basic analysis of both the constant scalar curvature and Sasaki-Einstein cases were combined in [BTF15] which also contains the foundational topological description. The current paper contains further results on the topology of the Sasaki-Einstein manifolds most of which appeared in [BTF13b], but were left out of [BTF15].

The main result concerning Sasaki-Einstein manifolds in [BTF15] is:

Theorem 1.1. Let $M_{l_1,l_2,w} = M \star_{l_1,l_2} S^3_w$ be the $S^3_w$-join with a regular Sasaki manifold $M$ which is an $S^1$-bundle over a compact positive Kähler-Einstein manifold $N$ with a primitive Kähler class $[\omega_N] \in H^2(N, \mathbb{Z})$. Assume that the relatively prime positive integers $(l_1, l_2)$ are

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the relative Fano indices given explicitly by
\[
l_1(\mathbf{w}) = \frac{I_N}{\gcd(w_1 + w_2, I_N)}, \quad l_2(\mathbf{w}) = \frac{w_1 + w_2}{\gcd(w_1 + w_2, I_N)},
\]
where \(I_N\) denotes the Fano index of \(N\). Then for each vector \(\mathbf{w} = (w_1, w_2) \in \mathbb{Z}^+ \times \mathbb{Z}^+\) with relatively prime components satisfying \(w_1 > w_2\) there exists a Reeb vector field \(\xi_v\) in the 2-dimensional \(\mathbf{w}\)-Sasaki cone on \(M_{l_1, l_2, w}\) such that the corresponding Sasakian structure \(\mathcal{S} = (\xi_v, \eta_v, \Phi, g)\) is Sasaki-Einstein (SE).

The procedure involved taking a join of a regular Sasaki-Einstein manifold \(M\) with the weighted 3-sphere \(S^3_w\), that is, \(S^3\) with its standard contact structure, but with a weighted contact 1-form whose Reeb vector field generates rotations with generally different weights \(w_1, w_2\) for the two complex coordinates \(z_1, z_2\) of \(S^3 \subset \mathbb{C}^2\). We call this the \(S^3_w\)-join. By the \(\mathbf{w}\)-Sasaki cone we mean the two dimensional subcone of Sasaki cone induced by the Sasaki cone of \(S^3_w\). It is denoted by \(t^+_w\) and can be identified with the open first quadrant in \(\mathbb{R}^2\).

Most of the SE structures in Theorem 1.1 are irregular. Such structures have irreducible transverse holonomy \([HS12]\), implying there can be no generalization of the join procedure to the irregular case. We must deform within the Sasaki cone to obtain them. Furthermore, it follows from \([RT11, CS12]\) that constant scalar curvature Sasaki metrics (hence, SE) imply a certain K-semistability.

The SE metrics obtained from Theorem 1.1 were obtained earlier by physicists \([GHP03, GMSW04b, GMSW04a, CLPP05, MS05]\) working on the AdS/CFT correspondence. Their method, particularly that of \([GMSW04a]\), is very closely related to the Hamiltonian 2-form approach of \([ACG06]\) (cf. Section 4.3 of \([Spa11]\)). In fact Theorem 1.1 indicates that the physicist’s results fit naturally into our geometric construction. Furthermore, we showed in \([BTF15]\) that our geometric approach leads naturally to an algorithm for computing the cohomology ring of the \(2n + 3\)-manifolds. In the present paper we explicitly compute the cohomology ring of all such examples of SE manifolds in dimension 7 showing that there are a countably infinite number of distinct homotopy types of such manifolds. The case that \(M\) is a standard odd dimensional sphere was computed in \([BTF15]\), so here we give the cohomology rings of the joins when \(M\) is a circle bundle over one of the remaining del Pezzo surfaces. Explicitly, for \(N = \mathbb{CP}^1 \times \mathbb{CP}^1\) we have

**Theorem 1.2.** For each relatively prime pair \((w_1, w_2)\) of positive integers there exist Sasaki-Einstein metrics on the 7-manifolds \(M_{l_1, l_2, w}^7\)
with cohomology ring
\[ \mathbb{Z}[x, y, u, z]/(x^2, l_2(w)xy, w_1w_2l_1(w)y^2, z^2, u^2, zu, zx, ux, uy) \]
with \((l_1, l_2) = (2, |w|)\) if \(w\) is odd, or \((l_1, l_2) = (1, \frac{|w|}{2})\) if \(w\) is even, where \(x, y\) are 2-classes, and \(z, u\) are 5-classes.

It is well-known that when \(N\) is the blow-up of \(\mathbb{CP}^2\) at \(k\) generic points, namely \(N = \mathbb{CP}^2 \# k\mathbb{CP}^2\) there is a Kähler-Einstein metric precisely for \(k = 3, \ldots, 8\). Then our results give

Theorem 1.3. For each relatively prime pair \((w_1, w_2)\) of positive integers there exist Sasaki-Einstein metrics on the 7-manifolds \(M^7_{k, w}\) with cohomology ring

\[
H^q(M^7_{k, w}, \mathbb{Z}) \approx \begin{cases} 
\mathbb{Z} & \text{if } q = 0, 7; \\
\mathbb{Z}^{k+1} & \text{if } q = 2, 5; \\
\mathbb{Z}_{w_1+w_2} \times \mathbb{Z}_{w_1w_2} & \text{if } q = 4; \\
0 & \text{otherwise}, 
\end{cases}
\]

with the ring relations determined by \(\alpha_i \cup \alpha_j = 0, w_1w_2s^2 = 0, (w_1 + w_2)\alpha_i \cup s = 0\), and \(\alpha_i, s\) are the \(k + 1\) two classes with \(i = 1, \ldots, k\) where \(k = 3, \ldots, 8\). Furthermore, when \(4 \leq k \leq 8\) the local moduli space of Sasaki-Einstein metrics has real dimension \(4(k - 4)\).

Of particular interest is the join \(M^{2r+3}_w = S^{2r+1} \ast_{l_1, l_2} S^3_w\) of the standard odd dimensional sphere with the weighted \(S^3_w\) where

\[
(l_1, l_2) = \left( \frac{r+1}{\gcd(w_1 + w_2, r+1)}, \frac{w_1 + w_2}{\gcd(w_1 + w_2, r+1)} \right).
\]

By Theorem 4.5 of [BTF15] its cohomology ring is

\[
\mathbb{Z}[x, y]/(w_1w_2l_1(w)x^2, x^{r+1}, x^2y, y^2)
\]

where \(x, y\) are classes of degree 2 and \(2r + 1\), respectively. Let \(k\) be the length of the prime decomposition of \(w_1w_2\). Then for arbitrary \(r\) we show that there are \(2^{k-1}\) Sasaki-Einstein manifolds of the form \(M^{2r+3}_w\) with cohomology ring given by Equation (2). For the manifolds \(M^7_w\) of dimension 7 \((r = 2)\) much more is known about the topology. These are special cases of what are called generalized Witten spaces in [Esc05]. In particular, the homotopy type was given in [Kru97], while the homeomorphism and diffeomorphism type was given in [Esc05]. For our subclass admitting Sasaki-Einstein metrics we give necessary and sufficient conditions on \(w\) for homotopy equivalence when the order of \(H^4\) is odd in Proposition 3.7 below. Thus, we answer in the affirmative the existence of Einstein metrics on certain generalized Witten manifolds.
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2. The \( w \)-Sasaki cone when \( c_1(D) = 0 \)

In this section we describe some of the properties of the \( w \)-Sasaki cone when \( c_1(D) = 0 \), ending with some examples. Since we require that \( N \) be a positive Kähler-Einstein manifold, we have \( c_1(N) = I_N[\omega_N] \) where \( I_N \) is the Fano index. Recall from [BTF15] that the cohomological Einstein condition \( c_1(D_{l_1,l_2,w}) = 0 \) implies

Lemma 2.1. Necessary conditions for the Sasaki manifold \( M_{l_1,l_2,w} \) to admit a Sasaki-Einstein metric is that \( I_N > 0 \), and that

\[
l_2 = \frac{|w|}{\gcd(|w|, I_N)}, \quad l_1 = \frac{I_N}{\gcd(|w|, I_N)}.
\]

The integers \( l_1, l_2 \) in Lemma 2.1 were called relative Fano indices in [BG00]. For the remainder of the paper we assume that these integers take the values given by Lemma 2.1 unless explicitly stated otherwise. Note that the Fano index \( I_N \) of a Fano manifold of complex dimension \( r \) is bounded by \( r + 1 \), thus, \( l_1 \) is also bounded by \( r + 1 \). Moreover, \( I_N = r + 1 \) if and only if the universal cover of the regular Sasaki manifold \( M \) is the standard sphere \( S^{2r+1} \).

We shall make use of the following easily verified proposition for low values of \( I_N \).

Proposition 2.2. Let \( M \) be a regular Sasaki-Einstein manifold and consider the join \( M \star_{l_1,l_2} S^3_\omega \). Then

(1) If \( I_N = 1 \) then \( (l_1, l_2) = (1, |w|) \).

(2) If \( I_N = 2 \), then \( (l_1, l_2) = \begin{cases} (2, |w|) & \text{if } |w| \text{ is odd;} \\ (1, \frac{|w|}{2}) & \text{if } |w| \text{ is even.} \end{cases} \)

(3) If \( I_N = 3 \), then \( (l_1, l_2) = \begin{cases} (3, |w|) & \text{if } 3 \text{ does not divide } |w|; \\ (1, \frac{|w|}{3}) & \text{if } 3 \text{ divides } |w|. \end{cases} \)

A natural question that arises is whether the \( w \)-cone contains a regular Reeb vector field.

Proposition 2.3. Assume \( w \neq (1, 1) \) and let \( K = \gcd(I_N, |w|) \). Then there are exactly \( K - 1 \) different \( w \)-Sasaki cones that have a regular Reeb vector field. These are given by

\[
w = \left( \frac{K + n}{\gcd(K + n, K - n)}, \frac{K - n}{\gcd(K + n, K - n)} \right),
\]
where \( 1 \leq n < K \).

**Proof.** By Proposition 3.4 of [BTF15], a \( w \)-Sasaki cone contains a regular Reeb vector field if and only if there is \( n \in \mathbb{Z}^+ \) such that

\[
w_1 - w_2 = n \frac{w_1 + w_2}{\gcd(J_N, w_1 + w_2)}.
\]

Clearly, for a solution we must have \( n < \gcd(J_N, w_1 + w_2) \). Then we have a solution if and only if

\[(K - n)w_1 = (K + n)w_2 \tag{3}\]

for all \( 1 \leq n < K \). Since \( w_1 > w_2 \) and they are relatively prime we have the unique solution Equation (3) for each integer \( 1 \leq n < K \). □

We have an immediate corollary to Proposition 2.3:

**Corollary 2.4.** If \( J_N = 1 \) there are no regular Reeb vector fields in any \( w \)-Sasaki cone with \( w \neq (1, 1) \).

**Example 2.5.** Let us determine the \( w \)-joins with regular Reeb vector field for \( J_N = 2, 3 \). For example, if \( J_N = 2 \) for a solution to Equation (3) we must have \( K = 2 \) which gives \( n = 1 \) and \( w = (3, 1) \). This has as a consequence Corollary 2.7 below. Similarly if \( J_N = 3 \) we must have \( K = 3 \), which gives two solutions \( w = (2, 1) \) and \( w = (5, 1) \).

**Example 2.6.** Let \( p \) and \( q \) be relatively prime positive integers satisfying \( p > 1 \) and \( 1 \leq q < p \). Recall that the contact structures \( Y^{p,q} \) on \( S^2 \times S^3 \) were discovered in [GMSW04b], where it is shown that there is a unique Sasaki-Einstein metric in the Sasaki cone of each such \( Y^{p,q} \). These SE metrics are most often irregular. From the viewpoint of the present work, \( Y^{p,q} \) is a join \( M_{l_1, l_2, w} = M^3 \star_{l_1, l_2} S^3_w \) where \( N = S^2 \) with its standard (Fubini-Study) Kähler structure. Hence, \( J_N = 2 \). This example has been treated in more detail elsewhere [BPT14, Boy11, BTF15], so we shall be very brief here. As in [BTF15] we have using Lemma 2.1

\[
w = p + q, p - q), \quad l_1 = \gcd(p + q, p - q), \quad l_2 = p.
\]

It follows from Proposition 2.2 that there are two cases depending on whether \( |w| \) is odd or even. In the former case \( p = |w| \), and in the latter \( p = \frac{|w|}{2} \). From Example 2.5 we have

\[
(4) \quad w = \frac{(p + q, p - q)}{\gcd(p + q, p - q)}, \quad l_1 = \gcd(p + q, p - q), \quad l_2 = p.
\]

Unfortunately, the conventions are slightly different. In [BPT14, Boy11] the convention \( w_1 \leq w_2 \) is used; whereas, here and in [BTF15] the opposite convention, \( w_1 \geq w_2 \), is used.
Corollary 2.7. For $Y^{p,q}$ the $w$-Sasaki cone has a regular Reeb vector field if and only if $p = 2, q = 1$ or equivalently $w = (3, 1)$.

We remark that the quotient of $Y^{2,1}$ by the regular Reeb vector field is $\mathbb{CP}^2$ blown-up at a point; whereas, we have arrived at it from the $w$-Sasaki cone of an $S^1$ orbibundle over $S^2 \times \mathbb{CP}^1[3, 1]$.

3. The Topology of the Sasaki-Einstein manifolds

We briefly recall the method used in [BTF15] to prove Theorem 1.1. The idea is that if we know the differentials in the spectral sequence of the fibration

\[ M \longrightarrow N \longrightarrow BS^1, \]

we can use the commutative diagram of fibrations

\[ \begin{array}{ccc}
M \times S^3_w & \longrightarrow & M_{l_1, l_2, w} \longrightarrow BS^1 \\
\downarrow & & \downarrow \\
N \times B\mathbb{CP}^1[w] & \longrightarrow & BS^1 \times BS^1
\end{array} \]

(6)

to compute the cohomology ring of the join $M_{l_1, l_2, w}$. Here $BG$ is the classifying space of a group $G$ or Haefliger’s classifying space [Hae84] of an orbifold if $G$ is an orbifold.

3.1. Examples in General Dimension. In this section we mainly give partial topological results for some examples of general dimension.

3.1.1. $M$ is a Standard Sphere. The topology of the join when $M$ is a regular Sasakian sphere $S^{2r+1}$ was worked out in [BTF15] and further studied in [BTF14c]. We shall treat the 7-dimensional case in more detail in Section 3.2.1 below; however, before doing so we give the following result for $M_{w}^{2r+3} = S^{2r+1} \times_{l_1, l_2} S^3_w$ with $(l_1(w), l_2(w))$ satisfying Equations (1).

Lemma 3.1. If $H^4(M_{w}^{2r+3}, Z) = H^4(M_{w'}^{2r+3}, Z)$, then $w'_1 w'_2 = w_1 w_2$ and $l_1(w') = l_1(w)$.

Proof. The equality of the 4th cohomology groups together with the definition of $l_1$ imply

\[ w'_1 w'_2 l_1 \gcd(|w|, r + 1)^2 = w_1 w_2 \gcd(|w'|, r + 1)^2. \]

Set $g_w = \gcd(|w|, r + 1)$ and $g_{w'} = \gcd(|w'|, r + 1)$. Assume $g_{w'} > 1$. Since $\gcd(w'_1, w'_2) = 1$, $g_{w'}$ does not divide $w'_1 w'_2$. Thus, $g_{w'}^2$ divides $g_w^2$. Interchanging the roles of $w'$ and $w$ gives $g_{w'} = g_w$ which implies $l_1(w') = l_1(w)$, and hence, the lemma in the case that $g_{w'} > 1$. Now
assume $g_{w'} = 1$. Then we have $w_1 w_2 = w'_1 w'_2 g_w^2$ which implies that $g_w$ divides $w_1 w_2$. But then since $w_1, w_2$ are relatively prime, we must have $g_w = 1$. □

Let us set $W = w_1 w_2$, and write the prime decomposition of $W = w_1 w_2 = p_1^{a_1} \cdots p_k^{a_k}$ Let $P_k$ be the number of partitions of $W$ into the product $w_1 w_2$ of unordered relatively prime integers, including the pair $(w_1 w_2, 1)$. Then a counting argument gives $P_k = 2^{k-1}$. Once counted we then order the pair $w_1 > w_2$ as before. Let $P_W$ denote the set of $(2r + 3)$-manifolds $M_{2r+3}$ with isomorphic cohomology rings. Then Lemma 3.1 implies that the cardinality of $P_W$ is $P_k = 2^{k-1}$. This proves

Proposition 3.2. Let $k$ denote the length of the prime decomposition of $w_1 w_2$, then there are $2^{k-1}$ simply connected Sasaki-Einstein manifolds $M_{2r+3} = S^{2r+1} \star_{l_1, l_2} S^3_{w}$ of dimension $2r + 3$ with isomorphic cohomology rings such that $H^4$ has order $w_1 w_2 l_1 (w)^2$.

3.1.2. $M$ is a Rational Homology Sphere. If we replace the standard odd dimensional sphere by a rational homology sphere $V^{2r+1}$ with a regular Sasakian structure the computations in [BTF15] immediately give

Proposition 3.3. The rational cohomology ring of the $S^3_{w}$-join $V^{2r+1} \star_{l_1, l_2, w} S^3_{w}$ of a rational homology sphere $V^{2r+1}$ is

$$Q[x, y]/(x^2, y^2)$$

where $x, y$ are classes of degree $2$ and $2r + 1$, respectively. Here the $l_1, l_2$ are any positive integers satisfying $\gcd(l_2, w_1 w_2 l_1) = 1$.

Examples of rational homology spheres with regular Sasaki-Einstein metrics are given in [BGN02]. They are the Sasakian homogeneous Stiefel manifolds $V_2(\mathbb{R}^{2n+1})$ of 2-frames in $\mathbb{R}^{2n+1}$ and the 3-Sasakian homogeneous 11-manifold $G_2/Sp(1)$. Since we want the join to have a Sasaki-Einstein metric somewhere it the Sasaki cone, we require that the pair $(l_1, l_2)$ to be the relative Fano indices of Lemma 2.1.

Example 3.4. The Stiefel manifold $V_2(\mathbb{R}^{2n+1})$ of dimension $4n - 1$. It is a circle bundle over the odd complex quadric $Q_{2n-1}(\mathbb{C})$. Its Fano index $J$ is $2n - 1$. So the relative Fano indices are

$$l_1(w) = \frac{2n - 1}{\gcd(|w|, 2n - 1)}, \quad l_2(w) = \frac{|w|}{\gcd(|w|, 2n - 1)}.$$

The reason for the subscript + on $Sp(1)$ is that there are two non-conjugate $Sp(1)$ subgroups in the exceptional Lie group $G_2$ which we denote by the subscripts $\pm$. The quotient by $Sp(1)_+$ is equivalent to $V_2(\mathbb{R}^7)$. 

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Moreover, the cohomology of $V_2(\mathbb{R}^{2n+1})$ is
\[
H^p(V_2(\mathbb{R}^{2n+1}), \mathbb{Z}) \approx \begin{cases} 
\mathbb{Z} & \text{if } p = 0, 4n - 1; \\
\mathbb{Z}_2 & \text{if } p = 2n; \\
0 & \text{otherwise.}
\end{cases}
\]

From the long exact homotopy sequence and the commutative diagram one easily obtains the partial results for the join $M_{l_1, l_2, w}(V) = V_2(\mathbb{R}^{2n+1}) *_{l_1, l_2} S^3_w$ when $n > 2$, namely $M_{l_1, l_2, w}(V)$ is simply connected, $H^3(M_{l_1, l_2, w}(V), \mathbb{Z}) = H^5(M_{l_1, l_2, w}(V), \mathbb{Z}) = 0$, and
\begin{equation}
\pi_1(M_{l_1, l_2, w}(V)) = H^2(M_{l_1, l_2, w}(V), \mathbb{Z}) \approx \mathbb{Z}, \quad H^4(M_{l_1, l_2, w}(V), \mathbb{Z}) \approx \mathbb{Z}_{w_1 w_2 l_1^2}.
\end{equation}

Since the Stiefel manifolds $V_2(\mathbb{R}^{2n+1})$ are $S^1$-bundles over a complex quadric, they are special cases of the next example.

**Example 3.5.** The homogeneous 3-Sasakian 11-manifold $G_2/Sp(1)_+$. It is a rational homology sphere with a $Z_3$ in cohomological degrees 4 and 8. By Proposition 2.3 in [BG00] the Fano index $f$ associated to $G_2/Sp(1)_+$ is 3. Then by Proposition 2.2 we have Sasaki-Einstein metrics on the simply connected 13-manifolds, $G_2/Sp(1)_+ *_{3, |w|} S^3_w$ if 3 does not divide $|w|$, and $G_2/Sp(1)_+ *_{1, \frac{|w|}{3}} S^3_w$ if 3 divides $|w|$. These 13-manifolds are simply connected with $\pi_2 = \mathbb{Z}$ and torsion in $H^4$.

3.1.3. $M$ is the link of a Fermat hypersurface. The projective Fermat hypersurface $F_{d,n+1}$ of degree $d$ in $\mathbb{CP}^{n+1}$ is described in homogeneous coordinates by the equation
\begin{equation}
z_0^d + z_1^d + \cdots + z_{n+1}^d = 0.
\end{equation}

It is Fano when $d \leq n + 1$ with index $3_{F_{d,n+1}} = n + 2 - d$ when $d \leq n + 1$. Moreover, they have a Kähler-Einstein metric when $\frac{n+1}{2} \leq d \leq n + 1$. So for this range of $d$ the Sasakian circle bundle $S_{d,n+1}$ over $F_{d,n+1}$ has a Sasaki-Einstein metric [BG00]. Note that $F_{2,2n}$ is the complex quadric $Q_{2n-1} \subset \mathbb{CP}^{2n}$ and $S_{2,2n} = V_2(\mathbb{R}^{2n+1})$ described in Example 3.4 and they are endowed with KE and SE metrics, respectively although $d$ is outside of the range given above. The integral cohomology ring of $F_{d,n+1}$ is well understood [KWS0]. It is torsion free with $H_*(F_{d,n+1}, \mathbb{Z}) = H_*(\mathbb{P}^{n}, \mathbb{Z})$ except in the middle dimension $n$ where the $n$th cohomology group of $F_{d,n+1}$ is $\mathbb{Z}^{b_n}$ when $n$ is odd, and $\mathbb{Z}^{b_{n+1}}$ when $n$ is even, where
\[
b_n = (-1)^n \left( 1 + \frac{(1-d)^{n+2} - 1}{d} \right).
\]
Then if \( n > 4 \) we see as in Example \( 3.4 \) that the join \( M_{l_1,l_2,w} = S^3_{d,n+1} \star_{l_1,l_2} S^3_w \) is simply connected satisfying the conditions of Equation \( (8) \). In order that the join has a Sasaki-Einstein metric in its \( w \)-Sasaki cone, we must choose the relative Fano indices to be

\[
    l_1(w) = \frac{n + 2 - d}{\gcd(|w|, n + 2 - d)}, \quad l_2(w) = \frac{|w|}{\gcd(|w|, n + 2 - d)}
\]

with \( \frac{n+1}{2} \leq d \leq n + 1 \) or \( d = 2 \). For \( 2 < d < \frac{n+1}{2} \) it is unknown whether there is such an SE metric.

3.2. Examples in Dimension 7. We focus attention to dimension seven in which case \( N \) is a del Pezzo Surface, namely \( \mathbb{CP}^2 \), \( \mathbb{CP}^1 \times \mathbb{CP}^1 \), and \( \mathbb{CP}^2 \) blown-up at \( k \) generic points with \( 1 \leq k \leq 8 \). Then the \( S^3_w \)-join a Sasakian circle bundle over \( N \) will be a Sasaki 7-manifold.

3.2.1. \( M = S^5, N = \mathbb{CP}^2 \). For \( \mathbb{CP}^2 \) with its standard Fubini-Study Kählerian structure, we have \( J_N = 3 \). From Example \( 2.5 \) we see that we have a regular Reeb vector field in the \( w \)-Sasaki cone in precisely two cases, either \( w = (2, 1) \), or \( w = (5, 1) \). In the first case the relative Fano indices are \( (l_1, l_2) = (1, 1) \) while in the second case they are \( (l_1, l_2) = (1, 2) \). In the former case our 7-manifold \( M_{(2,1)} = S^5 \star_{1,1} S^3_{(2,1)} \) is an \( S^3 \)-bundle over \( \mathbb{CP}^2 \); whereas, in the latter case the 7-manifold \( M_{(5,1)} = S^5 \star_{1,2} S^3_{(5,1)} \) is an \( L(2; 5, 1) \) bundle over \( \mathbb{CP}^2 \). Moreover, it follows from standard lens space theory that \( L(2; 5, 1) \) is diffeomorphic to the real projective space \( \mathbb{RP}^3 \). For general \( w \) we have two cases by Proposition \( 2.2 \), \( H^4(M_w^7, \mathbb{Z}) = \mathbb{Z}_{w_1w_2} \) if 3 divides \( |w| \) and \( H^4(M_w^7, \mathbb{Z}) = \mathbb{Z}_{w_1w_2} \) if 3 does not divide \( |w| \). In both cases the cohomology ring is given by

\[
    \mathbb{Z}[x, y]/(w_1w_2|x|^2, x^3, x^2y, y^2)
\]

where \( x, y \) are classes of degree 2 and 5, respectively. Notice that since 3 must divide \( w_1 + w_2 \) in the first case and \( w_1w_2 \) are relatively prime, the cohomology rings are never isomorphic for the two different cases.

Remark 3.6. Let us make a brief remark about the homogeneous case \( w = (1, 1) \) with symmetry group \( SU(3) \times SU(2) \times U(1) \). There is a unique solution with a Sasaki-Einstein metric as shown in [BG00]. However, dropping both the Einstein and Sasakian conditions, Kreck and Stolz [KS88] gave a diffeomorphism and homeomorphism classification. Furthermore, using the results of [WZ90], they show that in certain cases each of the 28 diffeomorphism types admits an Einstein metric. If we drop the Einstein condition and allow contact bundles with non-trivial \( c_1 \) we can apply the classification results of [KS88] to the Sasakian case. This will be studied elsewhere.
For dimension 7 we see from Proposition 2.2 that if 3 divides $w_1 + w_2$ then the order $|H^4|$ is $W$. However, if 3 does not divide $w_1 + w_2$ then the order of $|H^4|$ is $9W$. So by Lemma 3.1 $\mathcal{P}_W$ splits into two cases, $\mathcal{P}_W^0$ if $W + 1$ is divisible by 3, and $\mathcal{P}_W^1$ if $W + 1$ is not divisible by 3. Of course, in either case the cardinality of $\mathcal{P}_W$ is $2^{k-1}$ where $k$ is the number of prime powers in the prime decomposition of $W$.

**Proposition 3.7.** Suppose the order of $H^4$ is odd. The elements $M^7_w$ and $M^7_{w'}$ in $\mathcal{P}_W^0$ are homotopy inequivalent if and only if either

$$(\frac{w_1' + w_2'}{3})^3 \equiv \pm (\frac{w_1 + w_2}{3})^3 \mod Z_W.$$ 

The elements $M^7_w$ and $M^7_{w'}$ in $\mathcal{P}_W^1$ are homotopy inequivalent if and only if

$$(w_1' + w_2')^3 \equiv \pm (w_1 + w_2)^3 \mod Z_{9W}.$$

**Proof.** For $r = 2$ consider the $E_6$ differential $d_6(\beta) = l_2(w)^3 s^3$ in the spectral sequence of Theorem 4.5 of [BTF15]. Since $l_2$ is relatively prime to $l_1(w)^3 w_1 w_2$, this takes values in the multiplicative group $Z_{l^2 W}^*$ of units in $Z_{l^2 W}$. Taking into account the choice of generators, it takes its values in $Z_{l^2 W}^*/\{\pm 1\}$. According to Theorem 5.1 of [Kru97], $M^7_w, M^7_{w'} \in \mathcal{P}_W$ are homotopy equivalent if and only if $l_2(w')^3 = l_2(w)^3$ in $Z_{l^2 W}^*/\{\pm 1\}$. Of course, this means that $l_2(w')^3 = \pm l_2(w)^3$ in $Z_{l^2 W}^*$. Note that the the other two conditions of Theorem 5.1 of [Kru97] are automatically satisfied in our case. □

Using a Maple program we have checked some examples for homotopy equivalence which appears to be quite sparse. So far we haven’t found any examples of a homotopy equivalence. However, we have not done a systematic computer search which we leave for future work.

**Example 3.8.** Our first example is an infinite sequence of pairs with the same cohomology ring. Set $W = 3p$ with $p$ an odd prime not equal to 3, which gives $P_k = 2$. Then for each odd prime $p \neq 3$ there are two manifolds in $\mathcal{P}_W$, namely $M^7_{(3p,1)}$ and $M^7_{(p,3)}$. The order of $H^4$ is $27p$.

We check the conditions of Proposition 3.7. We find

$$(3p + 1)^3 \equiv 9p + 1 \mod 27p, \quad (p + 3)^3 \equiv p^3 + 9p^2 + 27 \mod 27p.$$ 

First we look for integer solutions of $p^3 + 9p^2 - 9p + 26 \equiv 0 \mod 27p$. By the rational root test the solutions could only be $p = 2, 13, 26$ none of which are solutions. Next we check the second condition of Proposition 3.7, namely, $p^3 + 9p^2 + 9p + 28 \equiv 0 \mod 27p$. Again by the rational root test we find the only possibilities are $p = 2, 7, 14, 28$, from which s
we see that there are no solutions. Thus, we see that $M^7_{(3p,1)}$ and $M^7_{(p,3)}$ are not homotopy equivalent for any odd $p \neq 3$.

By the same arguments one can also show that the infinite sequence of pairs of the form $M^7_{(9p,1)}$ and $M^7_{(p,9)}$, with $p$ an odd prime relatively prime to 3, are never homotopy equivalent.

**Remark 3.9.** In Example 3.8 we do not need to have $p$ a prime, but we do need it to be relatively prime to 3. In this more general case, there will be more elements in $\mathcal{P}_1^w$. For example, if $p = 55$ we have $P_k = 4$ and the pair $(M^7_{(165,1)}, M^7_{(55,3)})$ has the same cohomology ring as $M^7_{(33,5)}$ and $M^7_{(15,11)}$. However, they are not homotopy equivalent to either member of the pair nor to each other.

**Example 3.10.** A somewhat more involved example is obtained by setting $W = 5 \cdot 7 \cdot 11 \cdot 17$. Here $P_k = 8$, so this gives eight 7-manifolds in $\mathcal{P}_0^w$, namely,

$$M^7_{(5545,1)}, M^7_{(1309,5)}, M^7_{(935,7)}, M^7_{(595,11)}, M^7_{(385,17)}, M^7_{(187,35)}, M^7_{(119,55)}, M^7_{(85,77)}.$$ 

One can check that these do not satisfy the conditions for homotopy equivalence of Proposition 3.7. So they are all homotopy inequivalent.

It is easy to get a necessary condition for homeomorphism.

**Proposition 3.11.** Suppose $w_1'w_2' = w_1w_2$ is odd and that $M^7_w$ and $M^7_{w'}$ are homeomorphic. Then in addition to the conditions of Proposition 3.7 we must have

$$2(w_1' + w_2')^2 \equiv 2(w_1 + w_2)^2 \mod 3w_1w_2.$$

**Proof.** This is because the first Pontrjagin class $p_1$ is actually a homeomorphism invariant. From Kruggel [Kru97] we see that if 3 does not divide $|w|$

$$p_1(M^7_w) \equiv 3|w|^2 - 9w_1^2 - 9w_2^2 \equiv -6|w|^2 \mod 9w_1w_2,$$

which implies the result in this case. If 3 divides $|w|$ we have

$$p_1(M^7_w) \equiv -6\left(\frac{|w|}{3}\right)^2 \mod w_1w_2$$

and this implies the same result. \hfill \Box

Note that Equations (11) and (12) both imply the third condition of Theorem 5.1 in [Kru97] holds in our case. To determine a full homeomorphism and diffeomorphism classification requires the Kreck-Stolz

\footnote{This appears to be a folklore result with no proof anywhere in the literature. It is stated without proof on page 2828 of [Kru97] and on page 31 of [KL05]. We thank Matthias Kreck for providing us with a proof that $p_1$ is a homeomorphism invariant.}
3.2.2. Imposing with \( n \) phic to \( Z \) homotopy sequences we have invariants \([KS88]\) of \([BGM94]\) are free quotients of \( S \) pairwise relatively prime positive integers (that the 3-Sasakian 7-manifolds in \([BGM94]\) are given by a triple of 7-manifolds of type \( S \)). Our manifolds whose cohomology rings are of this type were called \([BG99]\) for their cohomology rings have the same form. Seven dimensional manifolds whose cohomology rings are of this type were called \([Kru97]\) where \( \pi_4(SU(3)) \approx 0 \) whereas, \( \pi_4(S^5 \times S^3) \approx \mathbb{Z}_2 \).

\[ \square \]

3.2.2. \( M = S^2 \times S^3, N = \mathbb{C}P^1 \times \mathbb{C}P^1 \). Note that this is Example 3.1.3 with \( n = d = 2 \). We have \( J_N = 2 \), so there are two cases: \( |w| \) is odd implying \( l_2 = |w| \) and \( l_1 = 2 \); and \( |w| \) is even with \( l_2 = \frac{|w|}{2} \) and \( l_1 = 1 \). In both cases the smoothness condition \( \gcd(l_2, l_1 w_i) = 1 \) is satisfied. The \( E_2 \) term of the Leray-Serre spectral sequence of the top fibration of diagram \([6]\) is

\[ E_2^{p,q} = H^p(\mathbb{B}S^1, H^q(S^2 \times S^3 \times S^3_w, Z)) \approx \mathbb{Z}[s] \otimes \mathbb{Z}[\alpha]/(\alpha^2) \otimes \Lambda[\beta, \gamma], \]

which by the Leray-Serre Theorem converges to \( H^{p+q}(M_{t_1,t_2,w}, Z) \). Here \( \alpha \) is a 2-class and \( \beta, \gamma \) are 3-classes. From the bottom fibration in Diagram \([6]\) we have \( d_2(\beta) = \alpha \otimes s_1 \) and \( d_4(\gamma) = w_1 w_2 s_2^3 \). From the commutativity of diagram \([6]\) we have \( d_2(\beta) = l_2 s \) and \( d_4(\gamma_w) = w_1 w_2 l_1^2 s^2 \) which gives \( E_4^{3,0} \approx \mathbb{Z}_{w_1 w_2 l_1^2}, E_4^{0,3} \approx \mathbb{Z}, E_4^{2,2} \approx \mathbb{Z}_{l_2}, \) and \( E_4^{0,3} = 0 \). Then using Poincaré duality and universal coefficients we obtain

[[KS88], [Ese05], [Kru05]]; however, they are quite complicated and the classification requires computer programming which we leave for future work.
Proposition 3.13. In this case $M^7_{l_1,l_2,w}$ with either $(l_1, l_2) = (2, |w|)$ or $(1, \frac{|w|}{2})$ has the cohomology ring given by

$$H^r(M^7_{l_1,l_2,w}, \mathbb{Z}) = \mathbb{Z}[x, y, u, z]/(x^2, l_2xy, w_1w_2l_1^2y^2, z^2, Zu, zx, ux, uy)$$

where $x, y$ are 2-classes, and $z, u$ are 5-classes.

There is only one case with a regular Reeb vector field, and that is $w = (3, 1)$ in which case the relative Fano indices are $(1, 2)$. Then the 7-manifold is $(S^2 \times S^3) \star_{1,2} S^3_{(3,1)}$ can be realized as an $L(2; 3, 1) \approx \mathbb{RP}^3$ lens space bundle over $\mathbb{CP}^1 \times \mathbb{CP}^1$. Proposition 3.13 and Theorem 1.1 prove Theorem 1.2.

3.2.3. $M = k(S^2 \times S^3), N = \mathbb{CP}^2$ blown up at $k$ generic points with $k = 1, \ldots, 8$. Equivalently we write $N = N_k = \mathbb{CP}^2 \# k\mathbb{CP}^2$. All the Kähler structures have an extremal representative, but for $k = 1, 2$ they are not CSC. However, for $k = 3, \ldots, 8$ they are CSC, and hence, Kähler-Einstein. Notice that when $4 \leq k \leq 8$ the complex automorphism group has dimension 0, so the $w$-Sasaki cone is the entire Sasaki cone. Moreover, if $5 \leq k \leq 8$ the local moduli space has positive dimension, and we can choose any of the complex structures. By a theorem of Kobayashi and Ochiai [KO73] we have $\mathcal{J}_{N_k} = 1$ for all $k = 1, \ldots, 8$. So $l_1 = 1, l_2 = |w|$, and by Corollary 2.4 there are no regular Reeb vector fields in the $w$-Sasaki cone with $w \neq (1, 1)$. In particular, if $4 \leq k \leq 8$, there are no regular Reeb vector fields in the Sasaki cone. Generally, these are $L(|w|; w_1, w_2)$ lens space bundles over $N_k$. Of course, the case $w = (1, 1)$ is just an $S^1$-bundle over $N_k \times \mathbb{CP}^1$ with the product complex structure which is automatically regular. These were studied in [BG00]. Let $S_k$ denote the total space of the principal $S^1$-bundle over $N_k$ corresponding to the anticanonical line bundle $K^{-1}$ on $N_k$. By a well-known result of Smale $S_k$ is diffeomorphic to the $k$-fold connected sum $k(S^2 \times S^3)$. We consider the join $S_k \star_{1,|w|} S^3_w$. The case $w = (1, 1)$ was studied in [BG00] where it is shown to have a Sasaki-Einstein metric when $3 \leq k \leq 8$. Moreover, in this case we have determined the integral cohomology ring (see Theorem 5.4 of [BG00]). Here we generalize this result.

Proposition 3.14. The integral cohomology ring of the 7-manifolds $M^7_{k,w} = S_k \star_{1,|w|} S^3_w$ is given by

$$H^q(M^7_{k,w}, \mathbb{Z}) \approx \begin{cases} \mathbb{Z} & \text{if } q = 0, 7; \\ \mathbb{Z}^{k+1} & \text{if } q = 2, 5; \\ \mathbb{Z}_{|w|}^k \times \mathbb{Z}_{w_1w_2}^4 & \text{if } q = 4; \\ 0 & \text{if otherwise,} \end{cases}$$
with the ring relations determined by \( \alpha_i \cup \alpha_j = 0, w_1 w_2 s^2 = 0, |w| \alpha_i \cup s = 0 \), where \( \alpha, s \) are the \( k + 1 \) two classes with \( i = 1, \cdots k \).

Proof. As before the \( E_2 \) term of the Leray-Serre spectral sequence of the top fibration of diagram (6) is

\[
E_2^{p, q} = H^p(BS^1, H^q(S_k \times S^3_w, \mathbb{Z})) \approx \mathbb{Z}[s] \otimes \prod_i \Lambda[\alpha_i, \beta_i, \gamma] / \mathcal{I},
\]

where \( \alpha_i, \beta_j, \gamma \) have degrees 2, 3, 3, respectively, and \( \mathcal{I} \) is the ideal generated by the relations \( \alpha_i \cup \beta_i = \alpha_j \cup \beta_j, \alpha_i \cup \alpha_j = \beta_i \cup \beta_j = 0 \) for all \( i, j \), \( \alpha_i \cup \beta_j = 0 \) for \( i \neq j \) and \( \gamma^2 = 0 \).

Consider the lower product fibration of diagram (6). As in the previous case the first non-vanishing differential of the second factor is \( d_4 \), and as in that case \( d_4(\gamma) = w_1 w_2 s^2 \). For the first factor we know from Smale’s classification of simply connected spin 5-manifolds that \( S_k \) is diffeomorphic to the \( k \)-fold connected sum \( k(S^2 \times S^3) \). Moreover, since \( N = \mathbb{CP}^2 \# k \mathbb{CP}^2 \), the first factor fibration is

\[
k(S^2 \times S^3) \longrightarrow \mathbb{CP}^2 \# k \mathbb{CP}^2 \longrightarrow B S^1.
\]

Here the first non-vanishing differential is \( d_2(\beta_i) = |w| \alpha_i \otimes s \). Again from the commutativity of diagram (6) for the top fibration we have \( d_2(\beta_i) = |w| \alpha_i \otimes s \) at the \( E_2 \) level and \( d_4(\gamma) = w_1 w_2 s^2 \) at the \( E_4 \) level. One easily sees that the \( k + 1 \) 2-classes \( \alpha_i \in E_2^{2,0} \) and \( s \in E_2^{0,2} \) live to \( E_\infty \) and there is no torsion in degree 2. Moreover, there is nothing in degree 1, and the 3-classes \( \beta_i \in E_2^{3,0} \) and \( \gamma \in E_4^{3,0} \) die, so there is nothing in degree 3. However, there is torsion in degree 4, namely \( \mathbb{Z}_{|w|}^k \times \mathbb{Z}_{w_1 w_2} \). The remainder follows from Poincaré duality and dimensional considerations.

This generalizes Theorem 5.4 of [BG00] where the case \( w = (1, 1) \) is treated and together with Theorem 1.1 proves Theorem 1.3.

Remark 3.15. Since \( |w| \) and \( w_1 w_2 \) are relatively prime, \( H^4(M_{k,w}^7, \mathbb{Z}) \approx \mathbb{Z}_{|w|}^{k-1} \times \mathbb{Z}_{w_1 w_2 |w|} \). We can ask the question: when can \( M_{k,w}^7 \) and \( M_{k',w'}^7 \) have isomorphic cohomology rings? It is interesting and not difficult to see that there is only one possibility, namely \( M_{1,(3,2)}^7 \) and \( M_{1,(5,1)}^7 \) in which case \( H^4 \approx \mathbb{Z}_{30} \).

References


References


[KS88] M. Kreck and S. Stolz, A diffeomorphism classification of 7-dimensional homogeneous Einstein manifolds with $SU(3) \times SU(2) \times U(1)$-symmetry, Ann. of Math. (2) 127 (1988), no. 2, 373–388. MR 89c:57042


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