HIGHER CONNECTED MANIFOLDS WITH POSITIVE RICCI CURVATURE

CHARLES P. BOYER AND KRZYSZTOF GALICKI

Abstract. We prove the existence of Sasakian metrics with positive Ricci curvature on certain highly connected odd dimensional manifolds. In particular, we show that manifolds homeomorphic to the 2k-fold connected sum of $S^{2n-1} \times S^{2n}$ admit Sasakian metrics with positive Ricci curvature for all $k$. Furthermore, a formula for computing the diffeomorphism types is given and tables are presented for dimensions 7 and 11.

Introduction

An important problem in global Riemannian geometry is that of describing the class of manifolds that admit metrics of positive Ricci curvature. The only known obstructions for obtaining such metrics come from either the classical Myers theorem or the obstructions to the existence of positive scalar curvature which is fairly well understood (see the recent review article [RS01] for discussion and references). Given the lack of obstructions it seems most natural to develop techniques for proving the existence of positive Ricci curvature metrics. Over the years several methods for doing so have appeared. These include symmetry methods, bundle constructions, surgery theory, algebro-geometric techniques. We refer the reader to the recent papers [BGN03c, BGN03b, GZ02, ST04] for a discussion of the history and pertinent references. In [BGN03c, BGN03b] the authors with M. Nakamaye introduced a method for proving the existence of positive Ricci curvature on odd dimensional manifolds which relies on a transverse version of Yau’s famous proof of the Calabi conjecture. The odd dimensional manifolds to which this method has been applied are hypersurfaces of isolated singularities coming from weighted homogeneous polynomials. All such manifolds are what are sometimes called “highly connected”. So far our methods have been successfully applied mainly to rational homology spheres [BG02, BGN02b, BG06a], homotopy spheres [BGN03b, BGK05, BGKT05], and connected sums of $S^2 \times S^3$ [BGN03a, BGN02a, BG03b] (See also [Kol04b]). A recent review of our method can be found in [BG05b].

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The purpose of this note is to prove the existence of Sasakian metrics of positive Ricci curvature on certain odd dimensional highly connected smooth manifolds. Manifolds of dimension $2n$ or $2n+1$ that are $n-1$ connected are often referred to as highly connected manifolds. They are relatively tractable and there is a classification of such manifolds by C.T.C Wall [Wal62, Wal67] and his students [Bar65, Wil72] as well as [Cro01]. Our first result concerns dimension $4n-1$, where we prove

**Theorem 1:** Let $n \geq 2$ be an integer, then for each positive integer $k$ there exist Sasakian metrics with positive Ricci curvature in $D_n(k)$ of the $|bP_{4n}|$ diffeomorphism classes of the $(4n-1)$-manifolds $2k\#(S^{2n-1} \times S^{2n})$ that bound a parallelizable manifold, where the number $D_n(k)$ is determined by the explicit formula A.4 given in the Appendix. In particular, $D_n(1) = |bP_{4n}|$ for all $n \geq 2$, so $2\#(S^{2n-1} \times S^{2n})$ admits Sasakian metrics of positive Ricci curvature in every diffeomorphism class.

Here $k\#(M_1 \times M_2)$ denotes the $k$-fold connected sum of the manifold $M_1 \times M_2$. In the case of the connected sums of products of standard spheres, metrics of positive Ricci curvature have been constructed previously by Sha and Yang [SY91]. However, the existence of such metrics for the exotic differential structures appears to be new. In dimension $4n+1$ we prove a somewhat weaker result

**Theorem 2:** For each pair of positive integers $(n,k)$ there exists a $(2n-1)$-connected $(4n+1)$-manifold $K$ with $H_2(K,\mathbb{Z})$ free of rank $k$ which admits a Sasakian metric of positive Ricci curvature. Furthermore, $K$ is diffeomorphic to one of the manifolds

$$
\#k(S^{2n} \times S^{2n+1}) , \quad (k-1)(S^{2n} \times S^{2n+1})\#T , \quad \#k(S^{2n} \times S^{2n+1})\#\Sigma^{4n+1}
$$

where $T = T_1(S^{2n+1})$ is the unit tangent bundle of $S^{2n+1}$, and $\Sigma^{4n+1}$ is the Kervaire sphere. For $k = 1$ the manifolds

$$
S^{2n} \times S^{2n+1} , \quad (S^{2n} \times S^{2n+1})\#\Sigma^{4n+1} , \quad T
$$

all admit Sasakian metrics with positive Ricci curvature. If $n = 1,3$ then $\#k(S^{2n} \times S^{2n+1})$ admits a Sasakian metric with positive Ricci curvature for all $k$.

The result here for $n = 1$ was given previously in [BGN03c]. For highly connected rational homology spheres we have

**Theorem 3:** Every $(2n-2)$-connected $(4n-1)$-manifold that is the boundary of a parallelizable manifold whose homology group $H_{2n-1}(K,\mathbb{Z})$ is isomorphic to $\mathbb{Z}_3$ admits Sasakian metrics with positive Ricci curvature. There are precisely $2|bP_{4n}|$ such smooth manifolds.

There are two distinct non-homeomorphic topological manifolds in this theorem and they are distinguished by their linking form in $H_{2n-1}(K,\mathbb{Z}) \approx \mathbb{Z}_3$. Each topological manifold is comprised of $|bP_{4n}|$ distinct diffeomorphism types.
1. Highly Connected Manifolds

The most obvious subclass of highly connected manifolds are the homotopy spheres which were studied in detail in the seminal paper of Kervaire and Milnor [KM63]. We briefly summarize their results. Kervaire and Milnor defined an Abelian group $\Theta_n$ which consists of equivalence classes of homotopy spheres of dimension $n$ that are equivalent under oriented h-cobordism. By Smale’s famous h-cobordism theorem [Sma62, Mil65] this implies equivalence under oriented diffeomorphism. The group operation on $\Theta_n$ is connected sum. Now $\Theta_n$ has an important subgroup $bP_n+1$ which consists of equivalence classes of those homotopy spheres which are the boundary of a parallelizable manifold. It is the subgroup $bP_{2n}$ that is important for us in the present work. Kervaire and Milnor proved:

(i) $bP_{2m+1} = 0$.

(ii) $bP_{4m}$ ($m \geq 2$) is cyclic of order $2^{m-2}(2^{m-1} - 1)$ numerator $\left( \frac{4B_m}{m} \right)$, where $B_m$ is the $m$-th Bernoulli number. Thus, for example $|bP_8| = 28, |bP_{12}| = 992, |bP_{16}| = 8128, |bP_{20}| = 130, 816$.

(iii) $bP_{4m+2}$ is either 0 or $\mathbb{Z}_2$.

Determining which $bP_{4m+2}$ is $\{0\}$ and which is $\mathbb{Z}_2$ has proven to be difficult in general, and is still not completely understood. If $m \neq 2^i - 1$ for any $i \geq 3$, then Browder [Bro69] proved that $bP_{4m+2} = \mathbb{Z}_2$. However, $bP_{4m+2}$ is the identity for $m = 1, 3, 7, 15$, due to several people [MT67, BJM84]. See [Lan00] for a recent survey of results in this area and complete references. The answer is still unknown in the remaining cases. Using surgery Kervaire was the first to show that there is an exotic sphere in dimension 9. His construction works in all dimensions of the form $4m+1$, but as just discussed they are not always exotic.

In analogy with the Kervaire-Milnor group $bP_{2n}$, Durfee [Dur77] defined

**Definition 1.1:** For $n \geq 3$ let $BP_{2n}$ denote the Grothendieck group of diffeomorphism classes of closed oriented $(n-2)$-connected $(2n-1)$-manifolds that bound parallelizable manifolds.

As with $bP_{2n}$ group multiplication in $BP_{2n}$ is the connected sum operation, with the standard sphere $S^{2n-1}$ as a 2-sided identity. Furthermore, for the Kervaire and Milnor group $bP_{2n}$ becomes a subgroup of $BP_{2n}$ again by Smale’s h-cobordism theorem. Unless otherwise stated we shall herefore assume that $n \geq 3$. We are mainly interested in those highly connected manifolds that can be realized as links of isolated hypersurface singularities defined by weighted homogeneous polynomials, so we have

**Definition 1.2:** We denote by $WHP_{2n-1}$ the set of smooth closed $(2n-1)$ dimensional manifolds that can be realized as the link of an isolated hypersurface singularity of a weighted homogeneous polynomial in $\mathbb{C}^{n+1}$.

It is easy to see that elements of $WHP_{2n-1}$ enjoy some nice properties.
Theorem 1.3: Let $M \in \text{WHP}_{2n-1}$. Then

(i) $M$ is highly connected, that is, it is $(n-2)$-connected.

(ii) $M$ is the boundary of a compact $(n-1)$-connected parallelizable manifold $V$ of dimension $2n$ with $H_n(V, \mathbb{Z})$ free.

(iii) If $n$ is even the $(n-1)$st Betti number $b_{n-1}(M)$ is even.

(iv) If $n \geq 3$ then $M$ is a spin manifold.

Proof. Parts (i) and (ii) follow from the Milnor Fibration Theorem [Mil68].

By [BG01] $M$ admits a Sasakian structure, so its odd Betti numbers are even up to the middle dimension by a well known result of Fujitani and Blair-Goldberg (cf. [BG05a]) which proves (iii). Item (iv) is a consequence of the analysis of Proposition 2.6 of [BGN03c].

From (i) and (ii) of Theorem 1.3 one sees that $\text{WHP}_{2n-1}$ is a subset of $\text{BP}_{2n}$, and by (iii) of Theorem 1.3 it is a proper subset. This can be contrasted with $\text{bP}_{2n}$ which by a result of Brieskorn [Bri66] satisfies $\text{bP}_{2n} \cap \text{WHP}_{2n-1} = \text{bP}_{2n}$ if $n \geq 3$. Notice, however, that $\text{WHP}_{2n-1}$ is not generally a submonoid.

We now discuss invariants that distinguish elements of $\text{BP}_{2n}$. First, by Poincare duality the only non-vanishing homology groups occur in dimension $0, n, 2n - 1$. Moreover, $H_n(K, \mathbb{Z})$ is free and rank $H_n(K, \mathbb{Z}) = \text{rank} H_{n-1}(K, \mathbb{Z})$. Thus, our first invariant is the rank of $H_{n-1}(K, \mathbb{Z})$, so we define

$$(1.1) \quad \text{BP}_{2n}(k) = \{ K \in \text{BP}_{2n} \mid \text{rank} H_{n-1}(K, \mathbb{Z}) = k \}.$$ 

This provides $\text{BP}_{2n}$ with a grading, namely

$$(1.2) \quad \text{BP}_{2n} = \bigoplus_k \text{BP}_{2n}(k)$$

which is compatible with the monoid multiplication in the sense that

$$(1.3) \quad \times : \text{BP}_{2n}(k_1) \times \text{BP}_{2n}(k_2) \longrightarrow \text{BP}_{2n}(k_1 + k_2).$$

Note that $\text{BP}_{2n}(0)$ is the submonoid of highly connected rational homology spheres, and that $\text{BP}_{2n}(k)$ is a $\text{bP}_{2n}$-module. More generally if $J$ is a complete set of homeomorphism invariants for $\text{BP}_{2n}$, we let $\text{BP}_{2n}(J)$ denote the subset of $\text{BP}_{2n}$ corresponding to the invariants $J$. Then $\text{BP}_{2n}(J)$ is a homogeneous $\text{bP}_{2n}$-module.

The remaining known invariants [Wal67, Dur71, Dur77] are a linking form on the torsion subgroup of $H_{n-1}(K)$ and a quadratic invariant on the 2n-manifold whose boundary is $K$. The precise nature of these invariants depends on whether $n$ is even or odd. For the case $n$ even Durfee [Dur77] shows that for $n \geq 3$ and $n \neq 4, 8$ there is an exact sequence

$$(1.4) \quad 0 \longrightarrow \text{bP}_{2n} \longrightarrow \text{BP}_{2n} \xrightarrow{\Psi} \mathbb{Z} \oplus KQ(\mathbb{Z}) \longrightarrow 0,$$

1 This proposition is incorrectly stated in [BGN03c]. One should add to the hypothesis the condition that the basic first Chern class $c_1(F, \xi)$ be proportional to the basic class $[dn]_B$.}
Theorem 1.4: Let $M$ be a highly connected manifold in $BP_{4n}$ such that $H_{2n-1}(M,\mathbb{Z}) = \mathbb{Z}^k$. Then $M$ is diffeomorphic to $k\#(S^{2n-1} \times S^{2n})\#\Sigma^{4n-1}$ for some $\Sigma^{4n-1} \in BP_{4n}$.

Notice that by a well-known result of Fujitani and Blair-Goldberg (cf. [BG05a]) $k\#(S^{2n-1} \times S^{2n})\#\Sigma^{4n-1}$ can admit a Sasakian structure only if $k$ is even.

For the case $n$ odd the diffeomorphism classification was obtained by Wall [Wa2], but for our purposes, the presentation in [Dur71] is more convenient. Let $K \in BP_{2n}$ with $K = \partial V$, where $V$ can be taken as $(n-1)$-connected and parallelizable. In this case the key invariant is a $\mathbb{Z}_2$-quadratic form

$$\psi : H_n(V,\mathbb{Z})/2H_n(V,\mathbb{Z}) \rightarrow \mathbb{Z}_2$$

defined as follows: Let $X$ be an embedded $n$-sphere in $V$ that represents a non-trivial homology class in $H_n(V,\mathbb{Z})$, and let $[X]$ denote its image in $H_n(V,\mathbb{Z})/2H_n(V,\mathbb{Z})$. Then $\psi([X])$ is the characteristic class in the kernel $\ker(\pi_{n-1}(SO(n) \rightarrow \pi_{n-1}(SO)) \approx \mathbb{Z}_2$ of the normal bundle of $X$. Let $\text{rad } \psi$ be the radical of $\psi$, i.e., the subspace of the $\mathbb{Z}_2$-vector space $H_n(V,\mathbb{Z})/2H_n(V,\mathbb{Z})$ where $\psi$ is singular. Then Durfee [Dur71] (see also [DK75]) proves

Theorem 1.5: Let $K_i \in BP_{2n}$ for $i = 1, 2$ with $n \geq 3$ odd be boundaries of parallelizable $(n-1)$-connected $2n$ manifolds $V_i$ with $\mathbb{Z}_2$ quadratic forms $\psi_i$. Suppose that $H_{n-1}(K_i,\mathbb{Z}) \approx H_{n-1}(K_2,\mathbb{Z})$, then

(i) if $n = 3$ or 7, then $K_1$ and $K_2$ are diffeomorphic;
(ii) if the torsion subgroups of $H_{n-1}(K_i,\mathbb{Z})$ have odd order and $\psi_i|\text{rad } \psi_i = 0$ for $i = 1, 2$, then $K_1 \approx K_2\#(c(\psi_1) + c(\psi_2))\Sigma$, where $c$
is the Arf invariant and $\Sigma$ is the Kervaire sphere, i.e., the generator of $bP_{2n}$;
(iii) if the torsion subgroups of $H_{n-1}(K, \mathbb{Z})$ have odd order and
$\psi_i \mid \text{rad } \psi_i \neq 0$ for $i = 1, 2$, then $K_1 \approx K_2 \approx K_2 \neq \Sigma$.

It is convenient to define $WHP_{2n-1}(k) = WHP_{2n-1}(2k) \cap BP_{2n}(k)$. Then (iii) of
Theorem 1.3 implies $WHP_{4n-1}(2k) = \emptyset$, whereas we shall see that $WHP_{4n-1}(2k) \neq \emptyset$ as well as $WHP_{4n+1}(k) \neq \emptyset$ for all $k$. Recently, in the
case $n = 3$, Kollár [Kol05, Kol06a] has discovered strong restrictions on the
torsion subgroups of $H_2(K, \mathbb{Z})$ in order that $K$ admit a Sasakian structure
which implies that $WHP_6(0)$ is a proper subset of $BP_6(0)$. One certainly
expects these types of restrictions to persist in higher dimension as well.

2. Branched Covers and Periodicity

In this section we discuss some results of Durfee and Kaufmann [DK75]
concerning the periodicity of branched covers. Let $K \subset S^{2n+1}$ be a simple
fibered knot or link ($n \geq 1$), by which we mean an $(n-2)$ connected $(2n-1)$
eMBEDded submanifold of $S^{2n+1}$ for which the Milnor fibration theorem holds. If $F$ is the Milnor fiber of the fibration $\phi : S^{2n+1} - K \rightarrow S^1$ then the
monodromy map $h : H_n(F) \rightarrow H_n(F)$ is a fundamental invariant of the link
$K$. Let $K_k$ be a $k$-fold cyclic branched cover of $S^{2n+1}$ branched along $K$. Then
Durfee and Kaufmann [DK75] show that there is an exact sequence

$$(2.1) \quad H_n(F) \xrightarrow{1 + h + \cdots + h^{k-1}} H_n(F) \rightarrow H_n(K_k) \rightarrow 0.$$ 

So homologically $K_k$ is determined by the cokernel of the map $1 + h +
\cdots + h^{k-1}$. Now suppose that $K$ is a rational homology sphere and that
the monodromy map $h$ of $K$ has period $d$. Then since $1 - h$ is invertible,
$1 + h + \cdots + h^{d-1}$ is the zero map in 2.1, and this determines the homology
of $K_d$. Summarizing

**Lemma 2.1:** [Durfee-Kaufman] Let $K$ be a fibered knot in $S^{2n+1}$ which
is a rational homology sphere such that the monodromy map has period $d$.
Suppose further that $K_k$ is a $k$-fold cyclic cover of $S^{2n+1}$ branched along $K$.
Then

(i) $H_n(K_d) \approx H_n(F) \approx \mathbb{Z}^\mu$ where $\mu$ is the Milnor number of $K$.
(ii) $H_s(K_{k+d}) \approx H_s(K_k)$ for all $k > 0$.
(iii) $H_s(K_{d-k}) \approx H_s(K_k)$ for all $0 < k < d$.

Notice that (i) determines a large class of $n-1$ connected $2n+1$-manifolds
whose middle homology group $H_n$ is free, and in certain cases this determines
the manifold up to homeomorphism. Items (ii) and (iii) give a homological
periodicity.

Durfee and Kaufman also show that there are both homeomorphism and
diffeomorphism periodicities in the case that $n$ is odd and $n \neq 1, 3, 7$. In
particular in this case, when the link $K$ is a rational homology sphere whose
monodromy map has period $d$, $K_{k+d}$ is homeomorphic to $K_k$. To obtain the

diffeomorphism periodicity let $\sigma_k$ denote the signature of the intersection
form on the Milnor fiber $F_k$. Again assuming that $K$ is a rational homology
sphere and $h$ has periodicity $d$, one finds that $K_{k+d}$ is diffeomorphic to

$$K_k$$

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sphere and $h$ has periodicity $d$, one finds that $K_{k+d}$ is diffeomorphic to

$$K_k$$

where $\frac{d}{8} - 1$ denotes $\frac{d}{8}$ copies of the Milnor sphere $\Sigma$. Here

we state the slightly more general Theorem 6.4 of Durfee [Dur77]:

**Theorem 2.2:** For even $n \neq 2, 4, 8$ let $K_i$ be $(n - 2)$-connected manifolds

that bound parallelizable manifolds $V_i$, with $i = 1, 2$. Suppose that the qua-

dratic forms of $K_i$ are isomorphic and $H_{n-1}(K_1, \mathbb{Z}) \approx H_{n-1}(K_2, \mathbb{Z})$. Then

$\sigma(V_2) - \sigma(V_1)$ is divisible by 8, and $K_2$ is diffeomorphic to $K_1 \# \frac{1}{8}(\sigma(V_2) -

\sigma(V_1))\Sigma$ where $\sigma(V)$ is the Hirzebruch signature of $V$.

**Remark 2.1:** Theorem 6.4 of [Dur77] as well as Theorem 5.3 of [DK75]

exclude the cases $n = 4$ and 8. However, it follows from [Wil72] and [Cro01]

that the diffeomorphism classification still holds in these cases since the

links we are considering here have no element of even order in the torsion

subgroup of $H_{n-1}$ (In fact the torsion subgroup vanishes in the case above).

This remark also pertains to the discussion for Theorem 3 below.

### 3. Positive Ricci Curvature on Links

Recall [BGN03c] that a Sasakian structure $(\xi, \eta, \Phi, g)$ is positive if the ba-

sic Chern class $c_1(F_\xi)$ of the characteristic foliation $F_\xi$ is positive. The im-

portance of positive Sasakian structures comes from Theorem A of [BGN03c]

which states that they give rise to Sasakian metrics with positive Ricci cur-

vature. An important ingredient in the proof of this result is the ‘transverse

Yau theorem’ of El Kacimi-Alaoui [EKA90], or equivalently for the cases at

hand, the orbifold version of Yau’s theorem. Now there is a natural induced

Sasakian structure on the link of a hypersurface singularity of a weighted

homogeneous polynomial [BG01]. Combining this with ‘orbifold adjunction

theory’ [BG05a] we obtain

**Theorem 3.1:** Let $L_f$ be the link of an isolated hypersurface singularity

of a weighted homogeneous polynomial $f$ of degree $d$ and weight vector $w$.

Suppose further that $|w| - d > 0$. Then $L_f$ admits a Sasakian metric with

positive Ricci curvature.

It is a simple task to construct positive Sasakian structures on links by

increasing the dimension.

**Proposition 3.2:** Let $L_{f'}$ be the link of a weighted homogeneous polynomial

$f'(z_2, \cdots, z_n)$ in $n-1$ variables with weight vector $w'$ and degree $d'$. Assume

that the origin in $\mathbb{C}^{n-1}$ is the only singularity so that $L_{f'}$ is smooth. Consider

the weighted homogeneous polynomial

$$f = z_0^2 + z_1^2 + f'$$
of degree \( d = \text{lcm}(2, d') \). Then the link \( L_f \) admits a Sasakian structure with positive Ricci curvature and \( b_{n-1}(L_f) = b_{n-3}(L_f') \).

Proof. There are two cases. If \( d' \) is odd then the weight vector of \( f \) is \( w = (d', d', 2w') \), whereas, if \( d' \) is even, then \( w = (\frac{d'}{2}, \frac{d'}{2}, w') \). In the first case we have \( |w| - d = d' + d' + 2|w'| - 2d' = 2|w'| > 0 \), while in the second case \( |w| - d = \frac{d'}{2} + \frac{d'}{2} + |w'| - d' = |w'| > 0 \). In either case \( L_f \) admits a Sasakian metric with positive Ricci curvature by Theorem 3.1. To prove the Betti number relation we follow Milnor and Orlik [MO70] and consider the divisor \( \text{div} \Delta_f \) of the Alexander polynomial \( \Delta_f \) of \( f \). Defining the relatively prime pairs \((u_i, v_i)\) to be the reduced form of \( \frac{d}{w_i} \), that is, \( \frac{d}{w_i} = \frac{u_i}{v_i} \) using [MO70] we have

\[
div \Delta_f = \prod_i (\frac{\Lambda_{u_i}}{v_i} - 1) = (\Lambda_2 - 1)^2 \prod_{i'} (\frac{\Lambda_{u_{i'}}}{v_{i'}} - 1) = \prod_{i'} (\frac{\Lambda_{u_{i'}}}{v_{i'}} - 1) = \text{div} \Delta_{f'},
\]

and for \( i = 2, \ldots, n \) we have \( \frac{d}{w_i} = \frac{d'}{w_i'} \). So \( u_i = u_{i'} \) and \( v_i = v_{i'} \) for the same range of \( i \). The Betti number equality then follows from [MO70]. \( \square \)

We note that it is easy to see that the appearance of the two 2’s in \( f \) implies that the klt conditions used to imply the existence of Sasakian-Einstein metrics [BGK05, BG06a] cannot be satisfied. So we can say nothing about the existence of Sasakian-Einstein metrics on these links.

4. Proofs of Theorems 1, 2 and 3

The links that we need to prove Theorems 1-3 involve Brieskorn-Pham polynomials of the form

\[(4.1) \quad f_{p,q} = z_0^p + z_1^q + z_2^2 + \cdots + z_n^2.\]

The link associated with \( f_{p,q} \) is

\[L_{p,q} = \{ f_{p,q} = 0 \} \cap S^{2n+1}.\]

By Proposition 3.2 all such links admit Sasakian metrics with positive Ricci curvature. One can view \( L_{p,q} \) as a \( p \)-fold branched cover of \( S^{2n-1} \) branched over the link \( L_q \) defined by the polynomial

\[f_q = z_1^q + z_2^2 + \cdots + z_n^2.\]

Proof of Theorem 1. Here we need the link \( L_{2(2k+1), 2k+1} \), i.e. \( p = 2(2k + 1), q = 2k + 1 \), with \( n \) even (here \( n \) corresponds to \( 2n \) in the statement of the theorem). In this case the degree of \( L_{2(2k+1), 2k+1} \) is \( d = 2(2k + 1) \) which is the period of the monodromy map of the link \( L_{2k+1} \). Furthermore, \( L_{2k+1} \) is a homotopy sphere by the Brieskorn Graph Theorem [Bri66] or [BG05a]. Now the link \( L_{2(2k+1), 2k+1} \) is a \( 2(2k + 1) \) branched cover of \( S^{2n-1} \) branched over \( L_{2k+1} \), so by item (i) of Lemma 2.1, we have

\[(4.2) \quad H_{n-1}(L_{2(2k+1), 2k+1}, \mathbb{Z}) \approx H_n(L_{2(2k+1), 2k+1}, \mathbb{Z}) \approx \mathbb{Z}^\mu = \mathbb{Z}^{2k}.\]
Here $\mu$ is the Milnor number [Mil68] of the link $L_{2k+1}$ which is easily computed by the formula for Brieskorn polynomials, namely

$$\mu = \prod_{i=1}^{n} (a_i - 1) = (2k + 1 - 1) \cdot 1 \cdots 1 = 2k.$$  

**Remark 4.1:** Notice that the link $L_{2(2k+1),2k+1}$ can be obtained by iterating Proposition 3.2 beginning with the Brieskorn manifold $M(2(2k+1), 2k+1, 2)$ which is described in Example 1 (pg. 320) of [Mil75]. As discussed there it is the total space of the circle bundle with Chern number $-1$ over a Riemann surface of genus $k$.

It now follows from Theorem 1.4 that $L_{2(2k+1),2k+1}$ is diffeomorphic to $2k#(S^{n-1} \times S^{n})#\Sigma^{4n-1}$ for some $\Sigma^{4n-1} \in bP_{4n}$. (Here $n$ is as in the statement of the theorem.) We now use the periodicity results of Durfee and Kauffman to determine the diffeomorphism type. First we notice that Theorem 1.4 together with Theorem 4.5 of [DK75] imply that for every positive integer $i$ and every positive integer $k$, the link $L_{2i(2k+1),2k+1}$ is homeomorphic to the connected sum $2k#(S^{n-1} \times S^{n})$. The diffeomorphism types are determined by Theorem 2.2 (Theorem 6.4 of [Dur77], see also Theorem 5.3 of [DK75]) together with Remark 2.1. Let $F_{i,k}$ denote the Milnor fibre of the link $L_{2i(2k+1),2k+1}$ and $\sigma(F_{i,k})$ its Hirzebruch signature. Then Theorem 2.2 says that for each pair of positive integers $i, j$ there is a diffeomorphism

$$L_{2i(2k+1),2k+1} \approx \left( \frac{\sigma(F_{i,k}) - \sigma(F_{j,k})}{8} \right) # L_{2j(2k+1),2k+1},$$

where $l\Sigma$ denotes the connected sum of $l$ copies of the Milnor sphere, and a minus sign corresponds to reversing orientation. Actually this formula follows from a signature periodicity result of Neumann as stated in Theorem 5.2 of [DK75]. From Durfee’s theorem the difference in signatures is always divisible by 8, so this expression makes sense. Equation 4.3 can be iterated; so it is enough to consider the case $i = 2$ and $j = 1$. In order to determine how many distinct diffeomorphism types occur in 4.3, we need to compute the signature of the Milnor fibres. This is done in Appendix A. It is interesting to note that not all diffeomorphism types can be attained. This ends the proof of Theorem 1.

**Proof of Theorem 2.** Now we have $n$ odd (corresponding to $2n + 1$ in the statement of the theorem) and there are several cases. First we take $p = 2(2k + 1), q = 2k + 1$ as in the proof of Theorem 1. Again this leads to the link $L_{2(2k+1),2k+1}$ with free homology satisfying Equation 4.2 except now $n$ is odd. Next we consider $q = 2k$ in Equation 4.1. The link $L_{2k}$ of the Brieskorn-Pham polynomial $f_{2k} = z_1^{2k} + z_2 + \cdots + z_n^2$ is a rational homology sphere by the Brieskorn Graph Theorem. Furthermore, its monodromy map has period $2k$. Then choosing $p = 2k$ in Equation 4.1 the link $L_{2k,2k}$ is $2k$-fold branched cover over $S^{2n+1}$ branched over the rational homology sphere.
so by item (i) of Lemma 2.1, we have
\[ H_{n-1}(L_{2k,2k}, \mathbb{Z}) \approx H_n(L_{2k,2k}, \mathbb{Z}) \approx \mathbb{Z}^k = \mathbb{Z}^{2k-1}. \]

These two cases now give links whose middle homology groups are free of arbitrary positive rank. However, unlike the case for \( n \) even this does not determine the homeomorphism type unless \( n = 3,7 \) in which case there is a unique diffeomorphism class. Indeed Theorem 1.5 implies we need to compute the quadratic form \( \psi \), and this appears to be quite difficult in all but the simplest case. From Theorem 1.5 one can conclude [Dur71] that if \( M \in BP_{4n+2} \) with \( H_{2n}(M, \mathbb{Z}) \) free of rank one, then it is homeomorphic to \( S^{2n} \times S^{2n+1} \) or the unit tangent bundle \( T = T_1(S^{2n+1}) \). (Now \( n \) is as in the statement of the theorem). So the diffeomorphism types at most differ by an exotic Kervaire sphere.

Furthermore, \( S^{2n} \times S^{2n+1}, T \) and \((S^{2n} \times S^{2n+1})\#\Sigma^{4n+1}\) generate the torsion-free submonoid of \( BP_{4n+2} \), there being relations in the monoid, namely, \( T\#T = 2\#(S^{2n} \times S^{2n+1}) \) and \( T\#\Sigma^{4n+1} = T \). (Some further relations may exist depending on \( n \) such as \( T_1(S^3) \approx S^2 \times S^3 \)). This proves the first statement in Theorem 2.

To prove the second statement we follow Durfee and Kauffman and consider a slightly different Brieskorn-Pham polynomial, namely \( z_0^k + z_1^2 + \cdots + z_n^2 \). For \( k = 1 \) we get as before a link \( L_{2,2} \) whose middle homology group is free of rank one. Thus, it is diffeomorphic to one of the three generators above by (i) of Lemma 2.1. Now as \( k \) varies we have a homological periodicity by (ii) and (iii) of Lemma 2.1. Durfee and Kauffman show that there is an 8-fold diffeomorphism periodicity, and they compute the \( \psi \) invariant to show that
\[ L_{2,2} \approx T, \quad L_{4,2} \approx (S^{2n} \times S^{2n+1})\#\Sigma^{4n+1}, \]
\[ L_{6,2} \approx T\#\Sigma^{4n+1} \approx T, \quad L_{8,2} \approx S^{2n} \times S^{2n+1}. \]
This proves Theorem 2.

\textbf{Proof of Theorem 3.} This is essentially a corollary of Proposition 7.2 of [Dur77] where Durfee considers the link \( K_k \) of the Brieskorn-Pham polynomial \( z_0^k + z_1^3 + z_2^2 + \cdots + z_n^2 \) for even \( n \geq 4 \). He shows that \( H_n(K_2, \mathbb{Z}) \approx H_n(K_4, \mathbb{Z}) \approx \mathbb{Z}_3 \), but that \( K_2 \) and \( K_4 \) have inequivalent linking forms. Furthermore, \( K_{6l+2} \) is diffeomorphic to \( K_2\#(-1)^{\frac{n}{2}}\Sigma^{4n-1} \) and \( K_{6l+4} \) is diffeomorphic to \( K_4\#(-1)^{\frac{n}{2}}\Sigma^{4n-1} \) where \( \Sigma^{4n-1} = K_5 \) is the Milnor generator.

\textbf{Appendix A. Computing the Signature}

There are several known methods for computing the signature of the Milnor fibre \( F \) of a Brieskorn manifold in the case when \( n \) is odd. This was first accomplished for homotopy spheres by Brieskorn [Bri66] and developed further by Hirzebruch and Zagier [Hir67, HZ74]. Our discussion follows
that in [Hir67]. Let $\mathbf{a} \in (\mathbb{Z}^+)^{n+1}$ and write $\mathbf{a} = (a_0, \cdots, a_n)$. Consider the Brieskorn manifold $M_{\mathbf{a}}$ defined by the link
\[ \{ z_0^{a_0} + \cdots + z_n^{a_n} = 0 \} \cap S^{2n+1}. \]
The Milnor fibre $F_{\mathbf{a}}$ can be represented by the Brieskorn manifold
\[ \{ z \in \mathbb{C}^{n+1} \mid z_0^{a_0} + \cdots + z_n^{a_n} = 1 \}. \]
For $n$ even the Hirzebruch signature of $F_{\mathbf{a}}$ is given by the function
\[ t(\mathbf{a}) = \# \{ x \in \mathbb{Z}^{n+1} \mid 0 < x_k < a_k \text{ and } 0 < \sum_{j=0}^{n} \frac{x_k}{a_k} < 1 \mod 2 \} \]
\[ -\# \{ x \in \mathbb{Z}^{n+1} \mid 0 < x_k < a_k \text{ and } 1 < \sum_{j=0}^{n} \frac{x_k}{a_k} < 2 \mod 2 \}. \]
(A.1)

Using methods of Fourier analysis, Zagier has obtain the following formula for $t(\mathbf{a})$:
\[ t(\mathbf{a}) = \frac{(-1)^{\frac{N-1}{2}}}{N} \sum_{j=0}^{N-1} \frac{\pi(2j+1)}{2\pi} \cot \frac{\pi(2j+1)}{2a_0} \cdots \cot \frac{\pi(2j+1)}{2a_n}, \]
where $N$ is any common multiple of the $a_i$'s.

We now adapt this formula to treat the link of the Brieskorn-Pham polynomial of Equation 4.1 with $N = 2(2k+1)$, namely, $\mathbf{a} = (2(2k+1), 2k+1, 2 \cdots, 2)$. Notice that we can always take the $N$ in Zagier’s formula A.2 to be the same as the $N$ in Equation 4.1. In this case we shall denote $t(\mathbf{a})$ by $t_d$ since the degree $d = 2(2k+1)$ is the periodicity as well. Likewise, we denote by $t_{2d}$ the signature $t(\mathbf{a})$ with $\mathbf{a} = (4(2k+1), 2k+1, 2 \cdots, 2)$. We find
\[ t_d = \frac{(-1)^{\frac{N}{2}}}{4k+2} \sum_{j=0}^{4k+1} (-1)^j \cot^2 \frac{\pi(2j+1)}{8k+4} \cot \frac{\pi(2j+1)}{4k+2}, \]
and
\[ t_{2d} = \frac{(-1)^{\frac{N}{2}}}{8k+4} \sum_{j=0}^{8k+3} (-1)^j \cot^2 \frac{\pi(2j+1)}{16k+8} \cot \frac{\pi(2j+1)}{4k+2}. \]

We want to compute $\tau_k = \frac{t_{2d} - t_d}{8}$. After some algebra we find that $(64k + 32)\tau_k$ equals
\[ \sum_{j=0}^{8k+3} (-1)^j \cot \frac{\pi(2j+1)}{16k+8} \left( \cot \frac{\pi(2j+1)}{16k+8} - \cot \frac{\pi(2j+1)}{8k+4} \right) \cot \frac{\pi(2j+1)}{4k+2}. \]
(A.3)

Now $\tau_k$ is always an integer, and by A.3 it is independent of $n$. We now define
\[ D_n(k) = \frac{|bP_{4n}|}{\gcd(\tau_k, |bP_{4n}|)}. \]
(A.4)
By Equation 4.3, \( D_n(k) \) represents the number of distinct diffeomorphism types that can be represented by our construction. Using MAPLE we give two tables consisting of a list of \( k \) and \( D_n(k) \) together with the ratio
\[
\frac{D_2(k)}{|bP_k|} = \frac{1}{\gcd(\tau_k, |bP_{4n}|)}
\]
for both the 7-manifolds \( \#2k(S^3 \times S^4) \) and the 11-manifolds \( \#2k(S^5 \times S^6) \) for various values of \( k \).

Notice that the prime factorization of \( |bP_{4n}| \) consists of high powers of two together with odd primes coming from the Bernoulli numbers. Since \( \tau_k \) is independent of \( n \), this gives rise to a bit of a pattern for the ratios \( \frac{D_2(k)}{|bP_k|} \). It is obvious that for \( k = 1 \) all possible diffeomorphism types occur, but this seems also to hold for \( k = 2 \). It is of course true whenever \( |bP_{4n}| \) is relatively prime to 3. If we look at the next case namely, \( bP_{16} \), we see that \( |bP_{16}| = 8128 = 2^6 \cdot 127 \). Comparing this with \( |bP_{12}| = 992 = 2^5 \cdot 31 \), we see that the same ratios will occur for the case \( \#2k(S^7 \times S^8) \) as for \( \#2k(S^5 \times S^6) \) for \( k = 1, \ldots , 30 \). It is interesting to contemplate whether the above gaps in the diffeomorphism types occur as a consequence of our method or whether they indicate an honest obstruction to the existence of positive Sasakian structures. At this stage we have no way of knowing.

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| \( k \) | \( \tau_k \) | \( D_2(k) \) | \( \frac{D_2(k)}{|bP_k|} \) |
|---|---|---|---|
| 1 | 1 | 28 | 1 |
| 2 | 3 | 28 | 1 |
| 3 | 6 | 14 | 1 |
| 4 | 10 | 14 | 1 |
| 5 | 15 | 28 | 1 |
| 6 | 21 | 4 | 1 |
| 7 | 28 | 1 | 1 |
| 8 | 36 | 7 | 1 |
| 9 | 45 | 28 | 1 |
| 10 | 55 | 28 | 1 |
| 20 | 210 | 2 | 1 |
| 48 | 1176 | 1 | 1 |
| 50 | 1275 | 28 | 1 |
| 100 | 5050 | 14 | 1 |
| 496 | 123256 | 1 | 1 |
| 500 | 125250 | 14 | 1 |
### Table 2: \(2k\#(S^5 \times S^6)\)

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Department of Mathematics and Statistics, University of New Mexico, Albuquerque, N.M. 87131

E-mail address: cboyer@math.unm.edu

Department of Mathematics and Statistics, University of New Mexico, Albuquerque, N.M. 87131

E-mail address: galicki@math.unm.edu