Elementary proof of an adiabatic invariance for spins in a circular particle accelerator

H.S. Dumas, J.A. Ellison, G.H. Hoffstaetter

Abstract

In this article, we revisit the problem of spin-orbit motion in a circular particle accelerator. Starting from a standard mathematical model, we use asymptotic analysis to show adiabatic invariance of the angle between spin and the periodic polarization direction along the closed orbit. Our method is mathematically rigorous, and is significantly simpler than the multiphase averaging methods used in our previous work. We also explore passage through resonance phenomena in this context; again the mathematics is elementary and the resonance can be reduced to properties of Fresnel integrals.

Contents

1 Introduction

In modern particle accelerators, a high level of spin-polarization is often needed, and it should be relatively stable over time. But to obtain high polarization levels at useful energies, particles must first be accelerated from low energy while maintaining most of their initial polarization. It is for this reason that adiabatic invariance of spin is important. In this article, we present a new, simpler approach to proving this adiabatic invariance in a standard mathematical model for spin-orbit motion.

On the closed orbit of a circular accelerator, the T-BMT equation for spin, S, may be written

$$\frac{d}{d\theta} S = A(\theta; r) S\,, \qquad A^T = -A\,, \qquad A(\theta + 2\pi; r) = A(\theta; r)\,, \tag{1}$$

where S is a 3-vector, A is a 3×3 matrix, the independent variable θ represents the azimuthal location in the accelerator (the interval $[0, 2\pi]$ corresponds to one turn around the ring), and r is a scalar parameter. This system is well understood, for fixed r, in terms of the

Floquet Theorem. *The principal solution matrix (PSM), $\Phi(\theta; r)$, for (1) can be written*

$$\Phi(\theta; r) = U(\theta; r) \exp(-\mathcal{J}\nu_0(r)\theta)\, U^T(0; r). \tag{2}$$

Here, $U^{-1} = U^T$, $U(\theta + 2\pi; r) = U(\theta; r)$, the eigenvalues of $\Phi(2\pi; r)$ are $\exp \pm i2\pi\nu_0(r)$ and 1, and the skew-symmetric matrix $\mathcal{J} = (0\ 1\ 0\ ; -1\ 0\ 0\ ; 0\ 0\ 0)$. For completeness we outline the proof of this in Appendix A.

Using the Floquet theorem it is easy to show the following facts.

1. Any two solutions S^1 and S^2 of (1) have the property that $S^1(\theta) \cdot S^2(\theta)$ is constant. Thus the Euclidean norm of a solution is a constant, as is the angle between any two solutions.
2. For each r, there is a 2π-periodic solution $n_0(\theta; r)$ with norm 1, and of course, from Fact 1, $S(\theta) \cdot n_0(\theta, r)$ is constant for every solution S.

Fact 1 follows easily by applying (2) to two different initial conditions. Fact 2 follows by applying the PSM to the initial condition $S_0 = U(0; r)(0, 0, 1)^T$ giving $n_0(\theta; r) = \Phi(\theta; r)S_0 = U(\theta; r)(0, 0, 1)^T$, i.e., n_0 is the third column of U.

It is important to understand what happens when r is slowly varying and that is the subject of this paper.

In our previous work [2], [3], we proved that if $r = \varepsilon\theta$, then for small values of ε, an adiabatic invariant exists. More precisely, let $\hat{S}$ evolve according to

$$\frac{d}{d\theta} \hat{S} = A(\theta; \varepsilon\theta)\hat{S}\,, \tag{3}$$

then $\hat{S}(\theta) \cdot n_0(\theta; \varepsilon\theta)$ is an adiabatic invariant. This was accomplished by applying a two-phase averaging theorem, due to Neistadt [REFERENCE?], to a transformed system using a Floquet-type transformation. Here we present a considerably simpler proof which is elementary, self-contained (in that it doesn't rely on any "big" theorems), and gives better and more explicit error bounds.

The Floquet transformation $S \to s$ via $S = U(\theta; r)s$ gives $\dot{s} = -\nu_0(r)\mathcal{J}s$ as is to be expected. The Floquet transformation $\hat{S} \to s$ via $\hat{S} = U(\theta; \varepsilon\theta)s$ gives

$$\frac{d}{d\theta}\begin{pmatrix} s_1 \\ s_2 \\ s_3 \end{pmatrix} = \begin{pmatrix} \varepsilon\left[\eta_3(\theta, \varepsilon\theta)\, s_2 - \eta_2(\theta, \varepsilon\theta)\, s_3\right] \ - \ \nu_0(\varepsilon\theta)\, s_2 \\ \varepsilon\left[\eta_1(\theta, \varepsilon\theta)\, s_3 - \eta_3(\theta, \varepsilon\theta)\, s_1\right] \ + \ \nu_0(\varepsilon\theta)\, s_1 \\ \varepsilon\left[\eta_2(\theta, \varepsilon\theta)\, s_1 - \eta_1(\theta, \varepsilon\theta)\, s_2\right] \end{pmatrix}, \qquad \mathbf{s}(0, \varepsilon) = \mathbf{s}^0. \tag{4}$$

Here the vector $\boldsymbol{\eta} = (\eta_1, \eta_2, \eta_3)^T$ is defined by the skew symmetric matrix $U^T D_2 U =: (0\ {-\eta_3}\ \eta_2\ ; \eta_3\ 0\ {-\eta_1}\ ; -\eta_2\ \eta_1\ 0\ )$. The derivation uses $D_1 U(\theta; r) - \nu_0(r)U(\theta; r)\mathcal{J} = A(\theta; r)U(\theta; r)$. It is easy to show that $\hat{S} \cdot n_0 = s_3$. Thus the conjecture (which we prove below) is that s_3 is an adiabatic invariant.

We note that Eqs. (4) are Eqs. (2.59) on p. 26 of [1] (see also Eq. (2) of [2] or Eq. (29) of [3], which are essentially the same).

2 Set-up and Statement of Results R1 and R2

We take the spin tune on the closed orbit to be[1]

$$\nu(\tau) = a + b\tau\,, \qquad a > 0\,, \quad b > 0\,, \tag{5}$$

where a and b are suitable positive constants, we assume the η_k are sufficiently smooth scalar functions of their arguments, and we assume ε is a sufficiently small parameter. For vectors we use the ordinary Euclidean norm $|\ |$ (e.g. $|\mathbf{s}| = \sqrt{s_1^2 + s_2^2 + s_3^2}$), along with the induced Euclidean norm $\|\ \|$ for matrices; a third norm is introduced for convenience below. It can be shown that $|\mathbf{s}|$ is constant, and we adopt the convention that $|\mathbf{s}| = |\mathbf{s}^0| = 1$. Finally, for our analysis, we gather the first two components of $\mathbf{s}$ into the column vector $\mathcal{S} = (s_1, s_2)^T$. We think of $\mathbf{s}$ as a vector in $\mathbb{R}^3$ and refer to s_3 as the vertical component and to $\mathcal{S}$ as the horizontal components.

In this article, we give three main results: R1, R2, and R3 (immediately below). R3 is the adiabatic invariance of s_3, and its proof uses R1 and R2. The $O(\sqrt{\varepsilon})$ error bound in R3 (rather than $O(\varepsilon)$) is due to a resonance phenomenon. In Section 5 we look at the resonance phenomenon in more detail and indicate how it could be used to refine the constant term in the error bound (but not the $O(\sqrt{\varepsilon})$ order of the error bound). (Revise this in terms of new §5) We also briefly discuss and summarize our results in Section 6. In proving the results, we use a key lemma in Section 3, and two propositions in Section 4 that are combined to give Result R3.

We begin by stating our first two results (R1 and R2) as follows:

R1. Adiabatic Invariance of the Norm of Horizontal Spin. *Consider the system (4) where $\nu(\tau) = a + b\tau$ and the η_k are sufficiently smooth functions of their arguments. Fix the constant $T > 0$. Then there is a positive constant $K = K(T)$ such that, for any $\varepsilon > 0$ and any initial condition $\mathbf{s}^0$ with $|\mathbf{s}^0| = 1$, we have $\Big||\mathcal{S}(\theta)| - |\mathcal{S}(0)|\Big| \le K\sqrt{\varepsilon}/\sqrt{b}$ for all $\theta \in [0, T/\varepsilon]$.*

We also have the following result about $(s_3(\theta))^2$, where we write $s_3(0) = s_3^0$, consistent with $\mathbf{s}(0) = \mathbf{s}^0$.

R2. Adiabatic Invariance of the Square of Vertical Spin. *Consider the system (4) where $\nu(\tau) = a + b\tau$ and the η_k are sufficiently smooth functions of their arguments. Fix the constant $T > 0$. Then for any $\varepsilon > 0$ and any initial condition $\mathbf{s}^0$ with $|\mathbf{s}^0| = 1$, we have $\Big|(s_3(\theta))^2 - (s_3^0)^2\Big| \le 2\,K\sqrt{\varepsilon}/\sqrt{b}$ for all $\theta \in [0, T/\varepsilon]$, where K is given in R1.*

R2 follows almost immediately from R1 (i.e., R2 is essentially a corollary of R1), as we now show:

Proof of R2 assuming R1 is true.

$$1 \;=\; |\mathbf{s}|^2 \;\; \text{(invariant)} \;=\; |\mathcal{S}(\theta)|^2 + (s_3(\theta))^2 \;=\; |\mathcal{S}(0)|^2 + (s_3^0)^2 \;\Longrightarrow$$

$$(s_3(\theta))^2 - (s_3^0)^2 \;=\; |\mathcal{S}(0)|^2 - |\mathcal{S}(\theta)|^2 \;=\; \Big(|\mathcal{S}(0)| + |\mathcal{S}(\theta)|\Big)\Big(|\mathcal{S}(0)| - |\mathcal{S}(\theta)|\Big) \;\Longrightarrow$$

$$\Big|(s_3(\theta))^2 - (s_3^0)^2\Big| \;\le\; 2\,\Big||\mathcal{S}(0)| - |\mathcal{S}(\theta)|\Big| \;\le\; 2\,K\sqrt{\varepsilon}/\sqrt{b} \qquad \text{for all} \qquad \theta \in [0, T/\varepsilon]. \;\; //$$

[1] ν is often called the "closed orbit spin tune."

Our main result is:

R3. Adiabatic Invariance of s_3. *Consider the system (4) where* $\nu(\tau) = a + b\tau$ *and the* η_k *are sufficiently smooth functions of their arguments. Fix the constant* $T > 0$. *Then there are positive constants* $C = C(T)$ *and* $\varepsilon_0 = \varepsilon_0(T)$ *such that, for any* $\varepsilon \in (0, \varepsilon_0]$ *and any initial condition* $\mathbf{s}^0$ *with* $|\mathbf{s}^0| = 1$, *we have* $|s_3(\theta) - s_3^0| \le C\sqrt{\varepsilon/b}$ *for all* $\theta \in [0, T/\varepsilon]$.

Remarks.

1. Using "big O" (or Landau) notation, a common way of abbreviating the conclusions of our results is to say that $|\mathcal{S}(\theta)| = |\mathcal{S}(0)| + O(\sqrt{\varepsilon})$ in R1, $(s_3(\theta))^2 = (s_3^0)^2 + O(\sqrt{\varepsilon})$ in R2, $s_3(\theta) = s_3^0 + O(\sqrt{\varepsilon})$ in R2; and all results hold on $O(1/\varepsilon)$ θ-intervals.

2. Physically, the results say that the spin vector $\mathbf{s}$ does not move far from its initial horizontal circle ($s_3 = s_3^0$, $r = r^0 = \sqrt{(s_1^0)^2 + (s_2^0)^2}$) over long θ-intervals, and that both the closeness (which is $O(\sqrt{\varepsilon})$) and the length of the θ-interval ($O(1/\varepsilon)$) are improved by taking smaller values of the parameter ε.

3. In our method, it is possible to use closed-orbit spin-tune functions $\nu = \nu(\tau)$ that are more general than $\nu(\tau) = a + b\,\tau$, but we treat the linear case here for simplicity and transparency.

4. We assume above that the η_k are "sufficiently smooth," and we use this assumption several times in the proof to justify the convergence of series, the boundedness of certain norms, etc. Although we do not try to find the weakest smoothness assumptions under which our result holds, we note here that it's enough to assume that the η_k are of class C^2 jointly in their arguments.

5. In the above, we assume that $b > 0$ so that the tune changes with ε and θ. If we allow $b = 0$ so the tune is fixed, our analysis leading to R1 may be easily modified to show that, for all $\theta \in [0, T/\varepsilon]$,

   (a) $\Big||\mathcal{S}(\theta)| - |\mathcal{S}(0)|\Big| \le K\,\varepsilon$ if a is not an integer, and

   (b) $\Big||\mathcal{S}(\theta)| - |\mathcal{S}(0)|\Big| \le K$ if a is an integer.

This behavior may be interpreted as a resonance phenomenon. In other words, if the tune $\nu = a$ is fixed and a not an integer, the system is "off resonance," giving the $O(\varepsilon)$ bound in (a) above. If a is an integer, the system is "on resonance," leading to a possible $O(1)$ change as in (b). Returning to $b > 0$, we have passage through resonance, giving the $O(\sqrt{\varepsilon/b})$ bound in R1. Further discussion of resonance is given in §5 below.

Jim, Are a) and b) proven in §5? Now they're proved at the end of (i) in Part 2 of §3. (Maybe it will be called Remark 6.) Also, bring in Appendix B.

3 Proof of adiabatic invariance

We assume the hypotheses of Result R1; i.e., we consider system (4), we assume that the η_k are sufficiently smooth, that $\nu(\tau) = a + b\tau$ $(a, b > 0)$, and we fix the constant $T > 0$. We divide the proof into two parts. In Part 1 we prove (13) assuming (11) is true, and in Part 2 we prove (11).

Proof of R1, PART 1.

We first transform the system to a form suitable for our method. We change the column vector $\mathcal{S} = (s_1, s_2)^T$ to the new column $\mathcal{W} = (w_1, w_2)^T$ using the transformation

$$\mathcal{S} = e^{G(\theta;\varepsilon)J}\mathcal{W}\,. \tag{6}$$

Here $J := \left[\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right]$ is the standard symplectic matrix, and $G'(\theta;\varepsilon) = g(\theta;\varepsilon) = -\nu + \varepsilon\,\eta_3$ with $G(0;\varepsilon) = 0$. In other words, $g(\theta;\varepsilon) = -a - b\,\varepsilon\,\theta + \varepsilon\,\eta_3(\theta,\varepsilon\theta)$ and $G(\theta;\varepsilon) = -a\,\theta - \frac{1}{2}b\,\varepsilon\,\theta^2 + \varepsilon\,N_3(\theta;\varepsilon)$, where $N_3' = \frac{d}{d\theta}N_3 = \eta_3$, and $N_3(0;\varepsilon) = 0$. Since (6) is a rotation, $|\mathcal{S}(\theta)| = |\mathcal{W}(\theta)|$.

Using (4), it is not difficult to check that $\mathcal{W}$ satisfies the differential equation (DE) $\mathcal{W}'(\theta) = \varepsilon\, e^{-G(\theta;\varepsilon)J}\eta(\theta,\varepsilon\theta)\, s_3(\theta)$, where $\eta = (-\eta_2, \eta_1)^T$. We write the solution of the DE in the integral form

$$\mathcal{W}(\theta) \;=\; \mathcal{W}(0) \;+\; \varepsilon \int_0^\theta e^{-G(\theta';\varepsilon)J}\eta(\theta',\varepsilon\theta')\, s_3(\theta')\, d\theta' \;=\; \mathcal{W}(0) \;+\; \varepsilon\,\mathcal{I}(\theta;\varepsilon)\,, \tag{7}$$

where the second equality defines $\mathcal{I}(\theta;\varepsilon)$. Using the periodicity of $\boldsymbol{\eta}(\theta;\tau)$ in θ, we expand the vector $\eta = (-\eta_2,\eta_1)^T$ in the vector Fourier series

$$\eta(\theta,\tau) = \sum_{n\in\mathbb{Z}} \hat{\eta}_n(\tau)\, e^{in\theta}, \tag{8}$$

and we note that because of the smoothness hypotheses, this series converges absolutely uniformly on any θ-interval of the form $[0, T/\varepsilon]$ $(\varepsilon > 0)$. Each vector Fourier coefficient $\hat{\eta}_n(\varepsilon\theta) = \hat{\eta}_n(\tau)$ is defined and uniformly bounded for all $\tau \in [0,T]$ and since η is real $\hat{\eta}_{-n} = \hat{\eta}_n^*$. (Note that the components of each $\hat{\eta}_n$ are $(-\hat{\eta}_{2n}, \hat{\eta}_{1n})^T$, where $\hat{\eta}_{kn}$ is the nth Fourier coefficient of η_k, $k = 1,2$.)

Using the uniform convergence of the Fourier series to exchange the order of integration and summation, we write $\mathcal{I}(\theta;\varepsilon)$ as the uniformly convergent series

$$\mathcal{I}(\theta;\varepsilon) \;=\; \sum_{n\in\mathbb{Z}} I_n(\theta;\varepsilon), \tag{9}$$

where

$$I_n(\theta;\varepsilon) \;=\; \int_0^\theta \left[e^{in\theta'} e^{[a\theta' + \frac{1}{2}b\varepsilon(\theta')^2]J}\right] \left[e^{-\varepsilon N_3(\theta',\varepsilon)J}\hat{\eta}_n(\varepsilon\theta')\, s_3(\theta')\right] d\theta' \;=\; \int_0^\theta u_n'\, v_n\, d\theta'. \tag{10}$$

Here u_n' is the first term in square brackets and v_n is the second. This way of writing I_n will be important in Part 2 where we show that v_n is slowly varying.

We now reduce our proof to showing the following inequality:

$$\sum_{n\in\mathbb{Z}} |I_n(\theta;\varepsilon)| \;\le\; \frac{K}{\sqrt{\varepsilon b}} \qquad \text{for all} \qquad \theta \in [0, T/\varepsilon]\,, \tag{11}$$

where $I_n = I_n(\theta;\varepsilon)$ is given by (9)–(10), and $K = K(T)$ is a constant independent of ε and b (K is given explicitly below at the end of Part 2). To see that R1 follows from (11), we use (7) and (9) to write $\mathcal{W}(\theta) - \mathcal{W}(0) = \varepsilon\,\mathcal{I}(\theta;\varepsilon) = \sum_{n\in\mathbb{Z}} I_n(\theta;\varepsilon)$, which, using (11), implies

$$|\mathcal{W}(\theta) - \mathcal{W}(0)| \;=\; \varepsilon\,|\mathcal{I}(\theta;\varepsilon)| \;\le\; \varepsilon \sum_{n\in\mathbb{Z}} |I_n(\theta;\varepsilon)| \;\le\; K\sqrt{\varepsilon}/\sqrt{b} \qquad \text{for all} \qquad \theta \in [0, T/\varepsilon]. \tag{12}$$

Now from (6), we have $\mathcal{S}(0) = \mathcal{W}(0)$, and since the transformation is a rotation, $|\mathcal{S}(\theta)| = |\mathcal{W}(\theta)|$, thus

$$\Big||\mathcal{S}(\theta)| - |\mathcal{S}(0)|\Big| \;=\; \Big||\mathcal{W}(\theta)| - |\mathcal{W}(0)|\Big| \;\le\; |\mathcal{W}(\theta) - \mathcal{W}(0)| \;\le\; K\sqrt{\varepsilon}/\sqrt{b} \qquad \text{for all} \qquad \theta \in [0, T/\varepsilon]. \tag{13}$$

(The first inequality is an example of the "reverse triangle inequality," which holds for any norm. This establishes R1, provided that inequality (11) is true, which we show below.

Proof of R1, PART 2.

In the second part, we complete the proof by showing that the estimate (11) holds.

We first introduce a few definitions and notational conventions. For convenience, given $T > 0$ and $\varepsilon > 0$, we define the θ-domain D by

$$D \;=\; [0, T/\varepsilon]. \tag{14}$$

Given a function $h = h(\theta;\varepsilon)$, we write

$$\|h\|_D = \sup_{\theta\in D} |h(\theta;\varepsilon)| \tag{15}$$

to denote the *uniform norm* of h over D. Although both D and the norm generally depend on ε, we suppress this dependence, viewing ε as a parameter (emphasized by the semicolon separating the arguments θ and ε). In the same spirit, we write $h' = h'(\theta;\varepsilon)$ to denote $\frac{d}{d\theta}h(\theta;\varepsilon)$.

We now state a simple but essential

Lemma. *Let $u = u(\theta;\varepsilon)$, $v = v(\theta;\varepsilon)$ be smooth, and set $I(\theta;\varepsilon) = \int_0^\theta u'(\theta';\varepsilon)\, v(\theta';\varepsilon)\, d\theta'$. Suppose there are positive constants c_1, c_2, c_3 such that for all $\varepsilon > 0$, we have $\|u\|_D \le c_1/\sqrt{\varepsilon}$, $\|v\|_D \le c_2$, and $\|v'\|_D \le c_3\,\varepsilon$.*

$$\text{Then for all } \varepsilon > 0, \qquad \|I\|_D \;\le\; \frac{k}{\sqrt{\varepsilon}}, \qquad \text{where} \qquad k = c_1\Big(2c_2 + c_3\,T\Big). \tag{16}$$

Proof. For $\varepsilon > 0$ and $\theta \in D$, we have $I(\theta;\varepsilon) = \int_0^\theta u'(\theta';\varepsilon)\, v(\theta';\varepsilon)\, d\theta' \;=\; u\,v\Big|_0^\theta \;-\; \int_0^\theta u\,v'\,d\theta' \;\Longrightarrow$ $|I(\theta;\varepsilon)| \;\le\; |u(\theta;\varepsilon)||v(\theta;\varepsilon)| \;+\; |u(0;\varepsilon)||v(0;\varepsilon)| \;+\; \theta\,\|u\|_D\|v'\|_D \;\le\; 2\|u\|_D\|v\|_D \;+\; \theta\,\|u\|_D\|v'\|_D \;\le$ $c_1\big(2c_2 + c_3\,\varepsilon\,\theta\big)/\sqrt{\varepsilon}$. Thus, for $\varepsilon > 0$, we have $\|I\|_D = \sup_{\theta\in D}|I(\theta;\varepsilon)| \le k/\sqrt{\varepsilon}$, with k as in (16). //

To apply the lemma, we first identify I, u and v (each indexed by n), then find the bounding constants c_1, c_2, and c_3 (also indexed by n).

We set

$I_n(\theta;\varepsilon) \;=\; \int_0^\theta u_n'(\theta';\varepsilon)\, v_n(\theta';\varepsilon)\, d\theta'$ (which is a column vector), with

$u_n(\theta;\varepsilon) \;=\; \int_0^\theta e^{in\theta'} e^{[a\theta' + \frac{1}{2}b\varepsilon(\theta')^2]J}\, d\theta'$ (a matrix), and

$v_n(\theta;\varepsilon) \;=\; e^{-\varepsilon N_3(\theta,\varepsilon)J}\,\hat{\eta}_n(\varepsilon\theta)\, s_3(\theta)$ (a column vector), so that I_n agrees with Eq. (10) above.

For the constants, we take

$c_{1n} \;=\; 16/\sqrt{b}$ (all n), why not below also?

$c_{2n} \;=\; \sup_{\tau\in[0,T]} |\hat{\eta}_n(\tau)|\,,$ and

$c_{3n} \;=\; \sup_{\tau\in[0,T]} |\hat{\eta}_n'(\tau)| \;+\; L \sup_{\tau\in[0,T]} |\hat{\eta}_n(\tau)|\,,$ where $L \;=\; \|\eta_1\|_D + \|\eta_2\|_D + \|\eta_3\|_D\,.$

It is not difficult to check that c_{1n}, c_{2n}, and c_{3n} fulfill the requirements of the lemma, as follows:

(i) We write $u_n(\theta;\varepsilon) = \int_0^\theta e^{in\theta'} e^{[a\theta' + \frac{1}{2}b\varepsilon(\theta')^2]J}\, d\theta'$, and we show $\|u_n\|_D \le c_{1n}/\sqrt{\varepsilon}$.

First we use the (easily checked) identity $e^{i\lambda}e^{\mu J} \;=\; e^{i(\lambda-\mu)}M \,+\, e^{i(\lambda+\mu)}M^*$, where λ, μ are real scalars, and M, M^* are the constant matrices $M = I + iJ$ and $M^* = I - iJ$, with $I := \left[\begin{smallmatrix}1&0\\0&1\end{smallmatrix}\right]$ and $J := \left[\begin{smallmatrix}0&1\\-1&0\end{smallmatrix}\right]$. (It's easy to check that the Euclidean matrix norms of I and J are $\|I\| = \|J\| = 1$.) Here we set $\lambda = n\theta'$ and $\mu = a\theta' + \frac{1}{2}b\varepsilon(\theta')^2$. Using these conventions, we write $u_n(\theta;\varepsilon) = \int_0^\theta e^{i\lambda}e^{\mu J}d\theta' = \int_0^\theta e^{i(\lambda-\mu)}\,d\theta' M + \int_0^\theta e^{i(\lambda+\mu)}\,d\theta' M^*$. We abbreviate the integrals by $u_n^-(\theta) := \int_0^\theta e^{i(\lambda-\mu)}\,d\theta'$ and $u_n^+(\theta) := \int_0^\theta e^{i(\lambda+\mu)}\,d\theta'$. The first integral u_n^- may be expressed as a multiple of a Fresnel integral by writing $u_n^-(\theta) = \int_0^\theta e^{i(\lambda-\mu)}\,d\theta' = \int_0^\theta e^{i[(n-a)\theta' - \frac{1}{2}b\varepsilon(\theta')^2]}d\theta' = \sqrt{2/(\varepsilon b)}\,e^{i(n-a)^2/(2\varepsilon b)}\int_{A_n}^{B_n(\theta)} e^{-ix^2}dx$, where we have used the change of dummy variable $x = (\varepsilon b\,\theta' - n + a)/\sqrt{2\varepsilon b}$, with the limits of integration $A_n = (a-n)/\sqrt{2\varepsilon b}$ and $B_n(\theta) = (\varepsilon b\,\theta - n + a)/\sqrt{2\varepsilon b}$. From the properties of Fresnel integrals, we know[2] that $\left|\int_A^B e^{-ix^2}dx\right| \le 2\sqrt{2}$ for any $A, B \in \mathbb{R}$. (This inequality is verified by noting that the associated Euler-Cornu spiral lies in a square with edges of length 2 in the complex plane, while the integral $\int_A^B e^{-ix^2}dx$ is equal to the [complex] difference between two corresponding points on the spiral. The modulus of the integral is therefore equal to the length of the line segment connecting these points, and so is bounded by the length of the longest line segment lying in the square.) Thus $\|u_n^-\|_D \le \sqrt{2/(\varepsilon b)}\,(2\sqrt{2}) = 4/\sqrt{\varepsilon b}$.

The second integral is treated in a similar way by writing $u_n^+(\theta) = \int_0^\theta e^{i(\lambda+\mu)}\,d\theta' = \int_0^\theta e^{i[(n+a)\theta' + \frac{1}{2}b\varepsilon(\theta')^2]}d\theta' = \sqrt{2/(\varepsilon b)}\,e^{-i(n+a)^2/(2\varepsilon b)}\int_{A_n^+}^{B_n^+(\theta)} e^{ix^2}dx$. The change of dummy variable here is $x = (\varepsilon b\,\theta' + n + a)/\sqrt{2\varepsilon b}$ leading to the new limits of integration $A_n^+ = (a+n)/\sqrt{2\varepsilon b}$ and $B_n^+(\theta) = (\varepsilon b\,\theta + n + a)/\sqrt{2\varepsilon b}$. This gives the same bound $\|u_n^+\|_D \le 4/\sqrt{\varepsilon b}$.] Finally, taking the uniform norm of u_n gives $\|u_n\|_D \le \|u_n^-\|_D\|M\| + \|u_n^+\|_D\|M^*\| = 2\|u_n^-\|_D\|M\| \le 16/\sqrt{\varepsilon b}$, where we have used the fact that the Euclidean matrix norms $\|M\| = \|M^*\| = 2$ (as is easily checked by computing the largest singular value of either matrix; then by a standard theorem of matrix theory, this is also the Euclidean norm of the matrix). Thus $\|u_n\|_D \le 16/\sqrt{\varepsilon b}$, giving $c_{1n} = 16/\sqrt{b}$ as needed.

Remark 6. FIRST ATTEMPT: Here we explain Remark 5 of §2. If $b = 0$ then $u_n^-(\theta) = \int_0^\theta e^{i[(n-a)\theta']}d\theta'$ and thus $\|u_n^-\|_D \le 2/|n-a|$ for $n \ne a$ and $\le T/\varepsilon$ for $n = a$, and $\|u_n^+\|_D \le 2/|n+a|$ for $n \ne -a$ and $\le T/\varepsilon$ for $n = -a$. Thus $\|u_n\|_D \le 4/|n-a| + 4/|n+a|$ when n is not an integer, and $\|u_n\|_D \le T/\varepsilon$ when n is an integer. Remark 5 follows.

DOES OUR QP AVERAGING WORK WHEN $b = 0$?

(ii) Next, we have $\|v_n\|_D = \|e^{-\varepsilon N_3 J}\hat{\eta}_n\,s_3\|_D \le \|\hat{\eta}_n\|_D = \sup_{\tau\in[0,T]}|\hat{\eta}_n(\tau)| = c_{2n}$. Here the inequality follows from $|s_3| \le 1$, and the second equality follows from $D = [0, T/\varepsilon]$, so that $\theta \in D \Rightarrow \varepsilon\theta = \tau \in [0, T]$.

(iii) Finally, we compute the θ-derivative (also using Eq. (4)): $v_n'(\theta;\varepsilon) = \varepsilon\,e^{\varepsilon N_3(\theta;\varepsilon)J}\Big(s_3\left[\hat{\eta}_n'(\varepsilon\theta) - i\,\eta_3(\theta;\varepsilon)\,J\,\hat{\eta}_n(\varepsilon\theta)\right] + \hat{\eta}_n(\varepsilon\theta)\left[\eta_2(\theta,\varepsilon\theta)\,s_1 - \eta_1(\theta,\varepsilon\theta)\,s_2\right]\Big)$. Taking the uniform norm of v_n' gives $\|v_n'\|_D \le \varepsilon\left[\|\hat{\eta}_n'\|_D + \|\hat{\eta}_n\|_D\left(\|\eta_1\|_D + \|\eta_2\|_D + \|\eta_3\|_D\right)\right] \le \varepsilon\left[\sup_{\tau\in[0,T]}|\hat{\eta}_n'(\tau)| + L\,\sup_{\tau\in[0,T]}|\hat{\eta}_n(\tau)|\right] = c_{3n}\,\varepsilon$. Here we've used $|\mathbf{s}| = 1 \Rightarrow |s_k| \le 1$ $(k = 1,2,3)$ and the fact that the uniform norms $\|\eta_k\|_D$ $(k = 1,2,3)$ are bounded because of the smoothness

[2] In §5 we improve on this estimate by looking more carefully at the resonance phenomenon, which, at the present stage, gives the worst-case $O(1/\sqrt{\varepsilon})$ estimate. Here we simply note that the estimate is improved when A and B have the same sign.

assumptions. We note that $\|v'\|_D \leq c_3\,\varepsilon$ makes our mention of "slowly varying" after (10) into a precise statement.

Having verified the hypotheses of the lemma for $\theta \in D$, we apply the lemma to conclude that $\|I_n\|_D \leq k_n/\sqrt{\varepsilon}$, where $k_n = c_{1n}(2\,c_{2n} + c_{3n}\,T) = 8(2\,c_{2n} + T c_{3n})/\sqrt{b}$. Using this, we verify inequality (11) by writing $\sum_{n\in\mathbb{Z}} |I_n(\theta;\varepsilon)| \leq \sum_n \|I_n\|_D \leq \sum_n k_n/\sqrt{\varepsilon} = \frac{8}{\sqrt{\varepsilon b}} \sum_n (2\,c_{2n} + T c_{3n}) = 8\Big[2S + TLS'\Big]\Big/\sqrt{\varepsilon b}$, where the sums $S = \sum_n \sup_{\tau\in[0,T]} |\hat{\eta}_n(\tau)|$ and $S' = \sum_n \sup_{\tau\in[0,T]} |\hat{\eta}'_n(\tau)|$ are convergent and finite by virtue of the smoothness hypotheses. Thus inequality (11) holds with $K = 8\Big[2S + TLS'\Big]$, and so Result R1 is proved (which also immediately implies Result R2, as shown in Section 2). //

4 Adiabatic invariance of s_3 (the vertical component of spin)

Here we prove R3. We treat the adiabatic invariance of s_3 here in a separate section because it is slightly different from our other results: the statement requires ε to be restricted to a small positive interval $(0, \varepsilon_0]$, and although the proof does make essential use of Result R2 from Section 2, it also requires us to combine two propositions (Propositions 1 and 2 below) in order to cover the entire domain of initial conditions $\mathbf{s}^0$.
Scott will investigate the b dependence in R3 [FIRST ATTEMPT COMPLETE, also checked once (briefly!)].

In order to prove R3, we first state two similar auxiliary results: one restricted to nonvertical initial spins, and one restricted to nonhorizontal initial spins. We then combine the two results (like overlapping charts in an atlas) to prove the statement above.

The first proposition treats initial spins lying in the northern hemisphere, away from the equator:

Proposition 1. Adiabatic Invariance of s_3 for Nonhorizontal Initial Spin. *Consider the system (4) where $\nu(\tau) = a + b\tau$ and the η_k are sufficiently smooth functions of their arguments. Fix the constants $T > 0$ and $0 < s_3^{min} < 1$. Then there are positive constants $C_1 = C_1(T, s_3^{min})$ and $\varepsilon_1 = \varepsilon_1(T, s_3^{min})$ such that, for any $\varepsilon \in (0, \varepsilon_1]$ and any initial condition $\mathbf{s}^0$ with $|\mathbf{s}^0| = 1$ and $s_3^0 > s_3^{min}$, we have $|s_3(\theta) - s_3^0| \leq C_1\sqrt{\varepsilon/b}$ for all $\theta \in [0, T/\varepsilon]$.*

The second proposition treats initial spins lying in the northern hemisphere, but away from the north pole:

Proposition 2. Adiabatic Invariance of s_3 for Nonvertical Initial Spin. *Consider the system (4) where $\nu(\tau) = a + b\tau$ and the η_k are sufficiently smooth functions of their arguments. Fix the constants $T > 0$ and $0 < s_3^{max} < 1$. Then there are positive constants $C_2 = C_2(T, s_3^{max})$ and $\varepsilon_2 = \varepsilon_2(T, s_3^{max})$ such that, for any $\varepsilon \in (0, \varepsilon_2]$ and any initial condition $\mathbf{s}^0$ with $|\mathbf{s}^0| = 1$ and $0 \leq s_3^0 < s_3^{max}$, we have $|s_3(\theta) - s_3^0| \leq C_2\sqrt{\varepsilon/b}$ for all $\theta \in [0, T/\varepsilon]$.*

It is a simple task to combine the two propositions to get the main result of this section, as we now show.

Proof of R3 from Propositions 1 and 2.

Assume the hypotheses of R3: Adiabatic Invariance of s_3 (see §2). Choose s_3^{min} from Proposition 1 and s_3^{max} from Proposition 2, both in the interval $(0,1)$ and such that $s_3^{min} < s_3^{max}$ (for example, $s_3^{min} = 1/2$ and $s_3^{max} = 2/3$). Set $C = \max\{C_1, C_2\}$ and $\varepsilon_0 = \min\{\varepsilon_1, \varepsilon_2\}$.

Now suppose $\varepsilon \in (0, \varepsilon_0]$ and consider any initial spin $\mathbf{s}^0$ with $|\mathbf{s}^0| = 1$. Clearly, by symmetry, it is enough to consider initial spins in the "northern hemisphere," with $s_3^0 \geq 0$. Given any such $\mathbf{s}^0$, either $s_3^0 > s_3^{min}$ or $s_3^0 < s_3^{max}$ (or both). In the first case, Proposition 1 applies, and gives the estimate $|s_3(\theta) - s_3^0| \leq C_1\sqrt{\varepsilon/b} \leq C\sqrt{\varepsilon/b}$; in the second case, Proposition 2 applies to give $|s_3(\theta) - s_3^0| \leq C_2\sqrt{\varepsilon/b} \leq C\sqrt{\varepsilon/b}$. In either case, we have $|s_3(\theta) - s_3^0| \leq C\sqrt{\varepsilon/b}$ for $\theta \in [0, T/\varepsilon]$, as required. //

We now give a detailed proof of Proposition 1 that relies on R2 from Section 2.

Proof of Proposition 1.

Assume the hypotheses of Proposition 1. We will show that the result holds with $C_1 = 2K/s_3^{min}$ and $\varepsilon_1 = (s_3^{min})^4 b/(4K^2)$, where $K = K(T)$ is the constant from our results in Section 2.

We consider two cases, depending on the choice of initial condition $\mathbf{s}^0$ with $s_3^0 > s_3^{min}$.

Case 1. The solution $\mathbf{s}$ never leaves the domain where $s_3 > 0$. In this case, the inequalities below in (18) clearly hold for all $\theta \in [0, T/\varepsilon]$, and Proposition 1 is established.

Case 2. There is a $\theta > 0$ such that $s_3(\theta) = 0$. In this case, we take $\theta_0 > 0$ to be the smallest such positive θ. (Thinking of θ as time, θ_0 could be called the "time of first exit from the domain $s_3 > 0$.") We then define the θ-domain E as

$$E = [0, \theta_0] \cap [0, T/\varepsilon]. \tag{17}$$

Observe that $\theta \in E \Longrightarrow |s_3(\theta) + s_3^0| \;\geq\; s_3^0 > s_3^{min}$.

Now using Result R2 (*Adiabatic invariance of the square of vertical spin*), we write, for all $\theta \in E$:

$$\begin{aligned} |s_3(\theta) - s_3^0|\,|s_3(\theta) + s_3^0| \;&=\; |(s_3(\theta))^2 - (s_3^0)^2| \;\leq\; 2\,K\sqrt{\varepsilon}/\sqrt{b} \quad \Longrightarrow \\ |s_3(\theta) - s_3^0| \;&\leq\; \frac{2\,K\sqrt{\varepsilon/b}}{|s_3(\theta) + s_3^0|} \;\leq\; \frac{2\,K\sqrt{\varepsilon/b}}{s_3^0} \;<\; \frac{2\,K}{s_3^{min}}\sqrt{\varepsilon/b} \;=\; C_1\sqrt{\varepsilon/b}\,. \end{aligned} \tag{18}$$

This shows that the desired estimate holds for $\theta \in E$. Finally, we show that the restriction $\varepsilon \in (0, \varepsilon_1]$ gives $E = [0, T/\varepsilon]$. Note (18) shows $|s_3(\theta) - s_3^0| < s_3^{min}$ for $\theta \in E$ and $\varepsilon \in (0, \varepsilon_1]$. Now assume $\theta_0 < T/\varepsilon$, so that $\theta_0 \in E$. Then $s_3^0 = |s_3(\theta_0) - s_3^0| < s_3^{min}$, which contradicts $s_3^0 > s_3^{min}$. Thus $\theta_0 \geq T/\varepsilon$, so $E = [0, T/\varepsilon]$ as needed.

Remark 7. FIRST ATTEMPT: Note that in the first line of (18) it is clear why the adiabatic invariance of the square of s_3 (our Result R2) is not sufficient to imply the adiabatic invariance of s_3 itself. Clearly, the term $|s_3(\theta) + s_3^0|$ can be arbitrarily small or even zero, and since it appears in the denominator of the second line of (18), it must be restricted away from zero. It is this fact that requires ε to be restricted to the interval $[0, \varepsilon_1]$, where $\varepsilon_1 > 0$ is the "threshold of validity" of Proposition 1. Maybe add that this keeps s_3 in the upper hemisphere for $[0, T/\varepsilon]$?

Brief discussion of the proof of Proposition 2.

We do not provide a detailed proof of Proposition 2, because it is similar in spirit to the proof of Proposition 1, and because it requires a reworking of the results of Sections 2 and 3 in a different coordinate system. More precisely, we begin with the original system (4), but instead of using the transformation (6), we introduce cylindrical coordinates (r, φ, s_3) via $s_1 = r\cos\varphi$ and $s_2 = r\sin\varphi$. We then follow arguments in that coordinate system analogous to those in Section 3. Because the cylindrical coordinates are singular at $r = 0$, we must exclude a circular cap of initial conditions near the north pole; however, this also allows us to include initial conditions near the equator, as needed.

5 Resonance Discussion

We have proven that s_3 is an AI, more precisely that $s_3(\theta) = s_{30} + O(\sqrt{\varepsilon/b})$ for $\theta \in [0, T/\varepsilon]$[3]. But why is the deviation from s_{30} of order $\sqrt{\varepsilon}$ and not ε^β with some other β? Why the mysterious 1/2? At one level the answer is that there is a "resonance" effect and a simple example is discussed in Appendix B. Our goal in this section is to expose the basic resonance phenomenon for (4) and an associated passage-through-resonance. In the process the reader will see ways to improve the order constant in R3. However, our goal here is a study of resonance and not a refinement of the order constant in the bounds.

The bound in R1 was the main ingredient in proving the AI of s_3 and it resulted from (13) which required an estimate on $\mathcal{W}(\theta) - \mathcal{W}(0) = \varepsilon\,\mathcal{I}(\theta;\varepsilon)$ from (7). Thus we want a bound on the growth of $\mathcal{I}$ for $\theta \in [0, T/\varepsilon]$. To estimate the growth of $\mathcal{I}$ we introduced the Fourier series for η in (8) and wrote $\mathcal{I} = \sum I_n$ in (9) - (10) reducing the problem to estimating I_n. Clearly $I_n = 0$ if $\hat{\boldsymbol{\eta}}_n = 0$ and in addition we found, using the Lemma , that

$$||I_n||_D \;\le\; ||u_n||_D\Big(2c_{2n} + c_{3n}\,T\Big)\,. \tag{19}$$

The c_{2n} and c_{3n} are defined just after the Lemma, they are even in n, depend only on $\hat{\boldsymbol{\eta}}_{\boldsymbol{n}}(\theta; r)$, and are zero if $\hat{\boldsymbol{\eta}}_n = 0$, so the bound in (19) is exact in that case. We then used the fact that $\left|\int_A^B e^{-ix^2}dx\right| \le 2\sqrt{2}$, for all $A, B \in \mathbb{R}$, to obtain the bound $||u_n||_D \le c_{1n}/\sqrt{\varepsilon} = 16/\sqrt{\varepsilon b}$. Thus $||I_n||_D \le (16/\sqrt{\varepsilon})RS_n$ where $RS_n = (2c_{2n} + c_{3n}\,T)/\sqrt{b}$ can be viewed as the resonance strength of the nth mode. The parameter b, as we shall see, can be viewed as the speed of passage thru resonance, and thus the slower the passage the bigger the resonance effect. It follows that I_n gives an $O(\sqrt{\varepsilon})$ contribution to $\mathcal{W} - \mathcal{W}_0$. This is suggestive of a passage-through-resonance phenomenon (see Appendix B), and we now take a closer look at u_n.

The bound $\left|\int_A^B e^{-ix^2}dx\right| \le 2\sqrt{2}$, that we used to bound u_n, can be improved by considering the actual values of A and B, and this will clarify the resonance phenomenon. Our goal in this section is to expose the basic passage-through-resonance phenomenon, refining the order constants in the bounds will be left to the interested readers.

Resonant effects occur in the I_n, thus the necessity of the Fourier expansion of $\boldsymbol{\eta}$. Of course, I_n and I_{-n} come in pairs so we rewrite (9) as

$$\mathcal{I}(\theta;\varepsilon) \;=\; I_0(\theta;\varepsilon) + \sum_{n>0}(I_n(\theta;\varepsilon) + I_{-n}(\theta;\varepsilon)) \tag{20}$$

and here we'll outline how bounds on $I_n + I_{-n}$ for $n > 0$ show a passage through resonance as θ varies. More precisely, we'll find no resonance behavior for $0 \le n < a$. For $n > a$ we'll find a resonance behavior with the center of the resonance passing at $\theta = \theta_r := (n - a)/b\varepsilon$. We leave the case where a

[3]T is a parameter to be chosen depending on the $\theta-$interval of interest.

is an integer to the reader. Recall our interval of interest for the dynamics is $0 \le \theta \le T/\varepsilon$, where T is chosen a priori, so there is no resonance in the interval of interest for n such that $n - a > bT$.

The resonance behavior in the $I_{\pm n}$ is contained in the $u_{\pm n}$ and integrating by parts as before gives

$$I_n(\theta) \;=\; u_n(\theta)v_n(\theta) - \int_0^\theta u_n(\theta')\, v_n'(\theta')\, d\theta' \tag{21}$$

where

$u_n(\theta;\varepsilon) \;=\; \int_0^\theta e^{in\theta'} e^{[a\theta' + \frac{1}{2}b\varepsilon(\theta')^2]J} d\theta'$, $\|v_n\|_D \le c_{2n}$, $\|v_n'\|_D \le c_{3n}$.

Thus we obtain

$$\|I_n(\theta)\| \;\le\; \|u_n(\theta)\| c_{2n} + \varepsilon c_{3n} \int_0^\theta \|u_n(\theta')\|\, d\theta' \;. \tag{22}$$

Also, looking back to Section 3, it can be seen that $u_n = u_n^- M + u_n^+ M^*$, $|u_n^-(\theta)| = 2\alpha | \int_{(a-n)\alpha}^{a-n+\varepsilon b\theta)\alpha} e^{ix^2} dx|$, where $\alpha = 1/\sqrt{2\varepsilon b}$, and $|u_n^+(\theta)| = |u_{-n}^-(\theta)|$, thus

$$\|u_n(\theta)\| \le |u_n^-(\theta)| 2 + |u_{-n}^-(\theta)| 2 \;. \tag{23}$$

For $\theta = 0$, $|u_n^-(\theta)| = |u_{-n}^-(\theta)| = 0$ and resonance occurs when 0 enters the domain of integration in their definition. This is the first indication that $a - n + \varepsilon b\theta = 0$ gives an interesting value of θ. Without loss of generality we can consider $n \ge 0$ in (23), and the two subcases $0 \le n < a$ and $0 < a < n$. We first note that an integration by parts gives

$$\left| \int_y^{y_1} e^{ix^2} dx \right| \;\le\; \frac{1}{y} \quad \text{for} \quad y_1 > y > 0 \;\; . \tag{24}$$

Case 1. $0 \le n < a$

Since $0 < a - n < a + n$, 0 is not in the integration domain for $u_n^{\pm}(\theta)$. Thus the bound in (24) can be used giving $|u_n^-(\theta)| \le 2/(a-n)$ and $|u_{-n}^-(\theta)| \le 2/(a+n)$. It follows that $|I_n(\theta)| \le 8a[c_{2n} + c_{3n}T]/(a^2 - n^2)$. By the symmetry in (23) and the fact that c_{2n} and c_{3n} are even in n, we have the same bound for $|I_{-n}(\theta)|$. Thus $I_n + I_{-n}$ gives an $O(1)$ contribution to $\mathcal{I}$.

Case 2. $0 < a < n$

Clearly $a + n > 0$, therefore using (24) we obtain $|u_{-n}^-(\theta)| = |u_n^+(\theta)| \le 2/(a+n) = O(1)$ for all θ and thus no resonant behavior. However, u_n^- goes thru a resonance at $\theta = \theta_r := (n-a)/\varepsilon b$ as we now discuss. For $\theta < \theta_r$, (24) gives $|u_n^-(\theta)| \le 2/(n - a - \varepsilon b\theta)$ which is $O(1)$ for θ not too close to θ_r. At $\theta = \theta_r$, $|u_n^-(\theta)| = 2\alpha \int_0^{(n-a)\alpha} e^{ix^2} dx \approx 2\alpha \int_0^\infty e^{ix^2} dx = \sqrt{\pi}\alpha = \sqrt{\pi/2\varepsilon b}$, for ε small, thus a growth from $O(1)$ to $O(1/\sqrt{\varepsilon b})$. At $\theta = 2\theta_r$, $|u_n^-(\theta)| = 4\alpha \int_0^{(n-a)\alpha} e^{ix^2} dx \approx 2\sqrt{\pi/2\varepsilon b}$. Thus we see the passage thru resonance starting from $O(1)$ to $\sqrt{\pi/2\varepsilon b}$, and then to $2\sqrt{\pi/2\varepsilon b}$ as θ grows from much less than θ_r to θ_r and then to $2\theta_r$.

Thus $I_n + I_{-n}$ gives an $O(1)$ contribution to $\mathcal{I}$ for θ less than but not too close to θ_r. As θ grows toward θ_r the contribution grows toward $O(1/\sqrt{\varepsilon b})$ and remains at that order for $\theta > \theta_r$. This is the passage through resonance.

RUTH

6 Discussion/comparison of results

Here we compare the results of this article with our earlier results in [2], [3], and we explain what motivated us to revisit the problem of adiabatic invariance of spin.

In Sections IIB and IIC of [3], we apply an averaging theorem of Neistadt [REF] to obtain the result that there exists a constant $c > 0$ such that $\sup_{\theta\in[0,1/\varepsilon]}|s_3(\theta) - s_3(0)| < c\sqrt{\varepsilon}$ for $b > 0$ and ε sufficiently small. If one accepts the averaging theorem as given (i.e., as a kind of "black box" not to be opened), then our proof in [3] is quite simple, consisting mainly of transforming the system into a form to which the averaging theorem applies.

However, this is unsatisfactory for several reasons. First, speaking generally, one might want to understand the averaging theorem used in our earlier proof (to "open the black box"), but this is a formidable task for the uninitiated, as can be verified by consulting [REF]. Second, and more specifically, one might want to use our result in a practical way, for example to check the dependence of the precision or duration of adiabatic invariance on the various parameters that enter the result when it is expressed physically. Again, this is a difficult or impractical undertaking, partly because it again requires an examination of the proof of the averaging theorem, but mostly because the averaging theorem is designed for very general systems, thus any attempt to trace the dependence of the result on parameters will find much of that dependence shrouded by that generality.

Our main motivation for writing the present article was to overcome the difficulties just mentioned, and we believe we have done so. Although our main result R3 is very similar to the result quoted from [3] above, the mathematics used here is significantly simpler than in [REF], and is self-contained (there are no "black boxes"). Perhaps more important is that the dependence on relevant parameters is explicit, since our new method is tailored to the system under study; in other words, nothing is lost to unneeded generality.

As an example of the simplicity and explicitness afforded by our new method, we examine the two important derived parameters in our main result R3: the bounding constant C and the threshold of validity ε_0. First, from the proof of R3, we have $C = \max\{C_1, C_2\}$, and although we don't explicitly compute C_2, it would be no more difficult to do so than to compute C_1, which we have done. Thus, in the case where $C_1 \geq C_2$ so that $C = C_1$, from the proof of Proposition 1 and Part (iii) of the proof of the Lemma, we find that

$$C = \frac{16\,(2S + TLS')}{s_3^{min}}\,,$$

where S and S' are the sums of the norms of the Fourier coefficients $\hat{\eta}_n$ and their derivatives $\hat{\eta}_n'$ (see Part (iii) above), T is the length parameter for the result's interval of validity, $L = \|\eta_1\|_D + \|\eta_2\|_D + \|\eta_3\|_D$, and s_3^{min} is the parameter chosen in Proposition 1.

Similarly, and again from the proof of R3, we have $\varepsilon_0 = \min\{\varepsilon_1, \varepsilon_2\}$, so in the case $\varepsilon_1 \leq \varepsilon_2$ with $\varepsilon_0 = \varepsilon_1$, again using the proof of Proposition 1 and Part (iii) of the proof of the Lemma, we get

$$\varepsilon_0 = \frac{b\,(s_3^{min})^4}{256\,(2S + TLS')^2}\,,$$

where b is the tune speed parameter from Eq. ( ), and the other constants are as above.

We point out that these formulas are quite simple and explicit, especially in comparison to whatever comparable expressions might be extracted from our earlier results based on general averaging theorems (we did not attempt those computations). We believe that the formulas ( ) and ( ) are simple enough to compare, at least in a qualitative way, with experimentally and heuristically derived results, and so could provide a means of checking the mathematical models from which they are derived. These simple expressions and the possibility of providing a self-contained elementary proof are the main reasons we revisited the problem of adiabaitc invariance of spin rigorously derived from a mathematical model.

7 Things left to do

1. Create Appendix A
2. Fix Appendix B

Appendix

A Floquet Derivation

B Resonance

A basic model of passage through a single resonance is the system $\dot{s}_3 = \varepsilon\cos(\psi - m\theta)$ and $\dot{\psi} = m - 1/2 + \varepsilon\theta$, since it begins well away from the resonance m and reaches it halfway through the interval $[0, 1/\varepsilon]$. The exact solution is $s_3(\theta) = s_3(0) + \varepsilon\int_0^\theta \cos((\varepsilon t^2 - t)/2 + \psi(0))\, dt$. Using a stationary phase argument, it is easy to show that this deviates from the solution of the averaged system $\bar{s}_3(\theta) = s_3(0)$ by $O(\sqrt{\varepsilon})$ at $O(1/\varepsilon)$ times.

The closely related system $\dot{s}_3 = \varepsilon\cos(\psi - m\theta)$ and $\dot{\psi} = \nu_0$, is either on resonance for $\nu_0 = m$ or off resonance if ν_0 is not an integer. On resonance the solution is $s_3(\theta) = s_3(0) + \varepsilon\theta\cos(\psi(0))$, which grows order one on $[0, 1/\varepsilon]$, so that s_3 is not an AI. Off resonance the exact solution is $s_3(\theta) = s_3(0) + \varepsilon(\sin[(\nu_0 - m)\theta + \psi(0)] - \sin(\psi_0))$, thus s_3 is an AI to $O(\varepsilon)$. ????

References

[1] G.H. Hoffstaetter, *High-Energy Polarized Proton Beams, A Modern View* (Springer Tracts in Modern Physics, Vol. 218), Springer Science+Business Media, Karlsruhe, 2006.

[2] G.H. Hoffstaetter, H.S. Dumas, and J.A. Ellison, Adiabatic invariants for spin-orbit motion, in *Proceedings of the Eighth European Particle Accelerator Conference* (Paris, France, June 3–7, 2002) 332–334, EPS-IGA & CERN, Geneva, 2002.

[3] G.H. Hoffstaetter, H.S. Dumas, and J.A. Ellison, Adiabatic invariance of spin-orbit motion in accelerators, *Phys. Rev. ST – Accelerators and Beams* **9** (1): 014001 [13 pages] (2006).

[4] D.P. Barber, J.A. Ellison and K. Heinemann, Quasiperiodic spin-orbit motion and spin tunes in storage rings, *Phys. Rev. ST – Accelerators and Beams* **7**, 124002 [33 pages] (2004).