Averaging for Quasiperiodic Systems with Applications

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Abstract

We present an improved averaging method for systems of ordinary differential equations with fast variables evolving quasiperiodically.

1 Introduction and basic problem

We consider

$$\frac{dx}{dt} = \varepsilon f(x,\theta), \qquad x(0) = \xi, \tag{1.1}$$

$$\frac{d\theta}{dt} = \omega, \qquad \theta(0) = \theta_0\,, \tag{1.2}$$

where $f : U \times \mathbb{R}^m \to \mathbb{R}^d$ is periodic with period 2π in each component of θ, $U \subset \mathbb{R}^d$ is open, $x, \xi \in U$, $\theta, \omega \in \mathbb{R}^m$ and $\varepsilon \geq 0$. For our analysis, we rewrite (1.1)–(1.2) in the nonautonomous form

$$\frac{dx}{dt} = \varepsilon f(x, \omega t + \theta_0)\,. \tag{1.3}$$

We introduce various smoothness conditions on f as needed below. We are interested in the behavior of Eq. (1.3) in open neighborhoods of 'frequency space' ($\mathbb{R}^m$) which are important in physical applications where the 'frequency vector' ω is not known exactly. The set of nonresonant ω is denoted by $\mathcal{N}$ and defined by $\mathcal{N} = \{\omega \in \mathbb{R}^m \,|\, k \in \mathbb{Z}^m,\ k\cdot\omega = 0 \implies k = 0\}$. Frequency vectors $\omega \notin \mathcal{N}$ are said to be resonant. We refer to (1.1)–(1.2) (or to (1.3)) as a quasiperiodic system. One standard definition characterizes $f(x, \omega t + \theta_0)$ as a quasiperiodic function whenever it is not periodic in t, so our usage contains this definition but also includes the periodic case.[1] We will work with the Fourier representation $f(x,\theta) \sim \sum_{k\in\mathbb{Z}^m} \hat{f}_k(x) \exp(ik\cdot\theta)$ where $\hat{f}_k(x) := (\frac{1}{2\pi})^m \int_{[0,2\pi]^m} f(x,\theta)\, \exp(-ik\cdot\theta)\, d\theta$ and where $k\cdot\theta$ denotes the scalar product of k and θ.

We explore approximate solutions based on normal forms for almost all ω and for $0 \leq t \leq T/\varepsilon$, where $T > 0$ is a parameter. Two important papers discussed this problem previously (Sáenz and Perko). We improve their results in the present paper, and illustrate our improvements with examples (FEL, spin, ??).

2 The basic comparison lemma (BCL)

In order to avoid the repetitious use of Gronwall estimates that naturally arise in our proofs, we present a lemma that compares the nearness of solutions for nearby systems once and for all. We consider the initial value problems (IVPs)

$$\frac{dx}{dt} = \varepsilon\, g(x,t), \qquad x(0,\varepsilon) = \xi \in U \tag{2.4}$$

$$\frac{du}{dt} = \varepsilon\, G(u, \varepsilon t), \qquad u(0,\varepsilon) = \xi\,. \tag{2.5}$$

Note that, by simple rescaling, the solution $u(t,\varepsilon)$ of (2.5) may be written as $u(t,\varepsilon) = v(\varepsilon t)$, where $v = v(\tau)$ is the solution of the ε-independent system

$$\frac{dv}{d\tau} = G(v,\tau), \qquad v(0) = \xi\,. \tag{2.6}$$

[1] In the Russian literature, a function of the form $f(x, \omega t + \theta_0)$ is said to be 'conditionally periodic in t,' and is quasiperiodic if it is not periodic in t. Unfortunately this terminology has not gained wide currency in the West.

We call v the guiding solution. We will determine the closeness of $x(t,\varepsilon)$ and $u(t,\varepsilon) = v(\varepsilon t)$ in terms of the closeness of g and G for t in certain long time intervals, and for ε sufficiently small (generally $\varepsilon \in [0, \varepsilon_0]$, where ε_0 is an appropriate "threshold").

For these systems, the open set $U \subset \mathbb{R}^d$ is generally the largest common domain of first arguments of g and G, and thus it is also the set of all possible initial conditions ξ. We now give a detailed set-up of elements used in the Basic Comparison Lemma.

2.1 Set-up for the BCL

(1) We assume that $g, G : U \times \mathbb{R} \to \mathbb{R}^d$ are continuous and are locally Lipschitz in their first arguments.

(To say $g : U \times \mathbb{R} \to \mathbb{R}^d$ is locally Lipschitz in its first argument means that, for any $x \in U$, there is a neighborhood N_x [with $x \in N_x \subset U$] and a nonnegative constant L_x such that for all $y_1, y_2 \in N_x$ and all $t \in [0, \infty)$, we have $|g(y_2,t) - g(y_1,t)| \le L_x |y_2 - y_1|$. [**Note uniformity in** t.] It can be shown that if g is locally Lipschitz on U, then it is Lipschitz on compact subsets of U. [REFERENCE—OR NOT?])

(2) For fixed initial condition $\xi \in U$, we choose an open set W containing ξ, with compact closure $\overline{W} \subset U$, and we take $L \ge 0$ to be the (smallest) x-Lipschitz constant for g on $\overline{W}$.

(3) We let $[0, \beta(\varepsilon))$ and $[0, \tau_{\max})$ denote the maximal forward time intervals of existence in W for the IVPs (2.4) and (2.6), respectively. (We note that $\beta(\varepsilon)$ and $\tau_{\max}$ depend on ξ; however, ξ will generally be fixed, so we suppress that dependence.)

(4) We choose a time constant $T \in (0, \tau_{\max})$ which will define the interval of comparison, and we let $I(\varepsilon, T) = [0, \beta(\varepsilon)) \cap [0, T/\varepsilon]$.

(5) For $t \in I(\varepsilon, T)$, we define $B(t,\varepsilon) := \left| \int_0^t \big(g(v(\varepsilon s), s) - G(v(\varepsilon s), \varepsilon s) \big)\, ds \right|$.

(6) We may want to define the bound M (appearing in proof of KBM below); then again, if it has nothing to do with the BCL, maybe not.

(7) Finally, we let $\widehat{B}(t,\varepsilon)$ be any function that is nondecreasing in t for each fixed ε such that $B(t,\varepsilon) \le \widehat{B}(t,\varepsilon)$. (Such $\widehat{B}(t,\varepsilon)$ exists since we can simply take $\widehat{B}(t,\varepsilon) = \sup_{0 \le \tau \le t} B(\tau, \varepsilon)$.)

2.2 The BCL

We may now state the

Basic comparison lemma (BCL). *Consider the set-up (1)–(7) for the IVPs (2.4), (2.5), (2.6), above. Let $x = x(t,\varepsilon)$, $u = u(t,\varepsilon) = v(\varepsilon t)$ be the solutions of (2.4), (2.5), and let ξ, W, L, M, T, $\beta(\varepsilon)$, $B(t,\varepsilon)$, and $\widehat{B}(t,\varepsilon)$ be as above.*

Then

(*a*) $$|x(t,\varepsilon) - v(\varepsilon t)| \;\le\; \varepsilon \widehat{B}(t,\varepsilon)\, e^{\varepsilon L t} \;\le\; \widehat{B}(T/\varepsilon, \varepsilon)\, e^{LT} \quad \textit{for} \quad t \in [0, \beta(\varepsilon)) \cap [0, T/\varepsilon]\,, \tag{2.7}$$

and

(*b*) *If $\widehat{B}(T/\varepsilon, \varepsilon) \to 0$ as $\varepsilon \to 0$, then there is an $\varepsilon_0 > 0$ such that whenever $\varepsilon \in [0, \varepsilon_0]$,*

$$|x(t,\varepsilon) - v(\varepsilon t)| \;\le\; \widehat{B}(T/\varepsilon, \varepsilon)\, e^{LT} \qquad \textit{for} \qquad t \in [0, T/\varepsilon]\,. \tag{2.8}$$

Proof. Clearly, for $t \in [0, \beta(\varepsilon)) \cap [0, T/\varepsilon]$, we have

$$|x(t,\varepsilon) - u(t,\varepsilon)| = \varepsilon \left| \int_0^t \big(g(x(s,\varepsilon),s) - g(u(s,\varepsilon),s) + g(u(s,\varepsilon),s) - G(u(s,\varepsilon),\varepsilon s) \big)\, ds \right| \tag{2.9}$$

$$\leq \ \varepsilon L \int_0^t |x(s,\varepsilon) - u(s,\varepsilon)|\, ds \ + \ \varepsilon \left| \int_0^t \big[g(u(s,\varepsilon),s) - G(u(s,\varepsilon),\varepsilon s) \big]\, ds \right|, \tag{2.10}$$

since $x(t,\varepsilon)$ and $u(t,\varepsilon) = v(\varepsilon t)$ remain in W for such t. Expressing the last inequality in terms of the guiding solution $v = v(\varepsilon t)$ and $B(t,\varepsilon)$, we have

$$|x(t,\varepsilon) - v(\varepsilon t)| \ \leq \ \varepsilon L \int_0^t |x(s,\varepsilon) - v(\varepsilon s)|\, ds \ + \ \varepsilon B(t,\varepsilon). \tag{2.11}$$

Since $B(t,\varepsilon) \leq \widehat{B}(t,\varepsilon)$ and $\widehat{B}$ is nondecreasing in t, an application of the generalized Gronwall inequality [REFERENCE? — OR NOT?] gives the first inequality in (2.7):

$$|x(t,\varepsilon) - v(\varepsilon t)| \ \leq \ \varepsilon \widehat{B}(t,\varepsilon)\, e^{\varepsilon L t} \qquad \text{for} \qquad t \in [0, \beta(\varepsilon)) \cap [0, T/\varepsilon]\,, \tag{2.12}$$

and the second inequality $\varepsilon\, \widehat{B}(t,\varepsilon)\, e^{\varepsilon L t} \ \leq \ \varepsilon\, \widehat{B}(T/\varepsilon,\varepsilon)\, e^{LT}$ follows from the fact that $\widehat{B}$ is nondecreasing in t. This establishes the first result (2.7).

Finally, under the assumption $\widehat{B}(T/\varepsilon,\varepsilon) \to 0$ as $\varepsilon \to 0$, by standard continuation arguments, there is an $\varepsilon_0 > 0$ such that $\beta(\varepsilon) > T/\varepsilon$ whenever $\varepsilon \in [0, \varepsilon_0]$. [DO WE WANT TO USE $o(1)$ NOTATION, INSTEAD OF $\to 0$?] Thus, for such ε, we have

$$|x(t,\varepsilon) - v(\varepsilon t)| \ \leq \ \widehat{B}(T/\varepsilon,\varepsilon)\, e^{LT} \qquad \text{for} \qquad t \in [0, T/\varepsilon]\,, \tag{2.13}$$

which is the second result (2.8). //

When using the BCL in a given problem, the main emphasis is on finding an appropriate normal form G and a nice estimate of $\widehat{B}(T/\varepsilon,\varepsilon)$ (at least $\widehat{B}(T/\varepsilon,\varepsilon) = o(1)$ as $\varepsilon \to 0$, and better in many cases).

3 Review of the general KBM result, and KBM for quasiperiodic systems

KBM stands for N.M. Krylov, N.N. Bogoliubov, and Y.A. Mitropolsky, who first developed and used the type of result we review here. [THIS MAY NOT BE QUITE RIGHT — CHECK SVM.] It's important to note that the KBM theorem applies to systems that are more general than our quasiperiodic system (1.1)–(1.2).

3.1 The general KBM result

Consider the systems (note G differs slightly but not essentially from (2.5) above):

$$\frac{dx}{dt} = \varepsilon\, g(x,t), \tag{3.14}$$

$$\frac{du}{dt} = \varepsilon\, G(u)\,. \tag{3.15}$$

We define $\delta(\varepsilon) = \varepsilon \sup_{(x,t)\in \overline{W}\times[0,T/\varepsilon]} \left\| \int_0^t \big(g(x,s) - G(x) \big)\, ds \right\|$ and recall $I(\varepsilon,T) = [0,\beta(\varepsilon)) \cap [0,T/\varepsilon]$.

Theorem 1 *Assume that systems (3.14), (3.15) are set up in accordance with items (1) through (7) in §2.1, and that* $\frac{1}{P}\int_0^P g(x,t)\,dt \to G(x)$ *as* $P \to \infty$ *uniformly for* $x \in \overline{W}$*. Then* $\delta(\varepsilon) \to 0$ *as* $\varepsilon \to 0$*, and there is a constant* $C(T) > 0$ *such that for all* $t \in I(\varepsilon, T)$*, we have*

$$|x(t,\varepsilon) - v(\varepsilon t)| \ \le\ C(T)\sqrt{\delta(\varepsilon)}. \tag{3.16}$$

Furthermore, there is an $\varepsilon_0 > 0$ *such that whenever* $\varepsilon \in [0, \varepsilon_0]$*, (3.16) holds for all* $t \in [0, T/\varepsilon]$*.*

Outline of Proof. The fact that $\delta(\varepsilon) \to 0$ as $\varepsilon \to 0$ follows from Sáenz' Lemma 1, or our Appendix below. From Sáenz' Lemma 2 we have $B(t,\varepsilon) \le 2t\sqrt{LM\delta(\varepsilon)} =: \widehat{B}(t,\varepsilon)$. The BCL then gives $|x(t,\varepsilon) - v(\varepsilon t)| \ \le\ \varepsilon\, \widehat{B}(t,\varepsilon)\, e^{\varepsilon L t} \ =\ 2\varepsilon t\sqrt{LM\delta(\varepsilon)}\, e^{\varepsilon L t} \ \le\ C\sqrt{\delta(\varepsilon)}$ for $t \in I(\varepsilon, T)$. Finally, by taking $\varepsilon_0 > 0$ small enough, and $\varepsilon \in [0, \varepsilon_0)$, we can make $\delta(\varepsilon)$ small enough to ensure that $x(t,\varepsilon)$ remains away from the boundary of W, so that (3.16) holds on $[0, T/\varepsilon]$. //

[DO WE WANT TO LOOK AT USING W VERSUS $\overline{W}$?]

3.2 KBM for quasiperiodic systems

In the quasiperiodic case, the general KBM result can be combined in a standard way with Diophantine ω s to show that $\delta(\varepsilon) = O(\varepsilon)$, in which case the error estimate improves from $o(1)$ to $O(\varepsilon^{1/2})$. Here we briefly describe results of this type, before improving them in two important ways below in §5.

We thus consider Theorem 1 in the special case where $g(x,t) = f(x, \omega t + \theta_0)$, and where f is as described in §1 and such that $C(f,\omega) = \sum_{k\cdot\omega \ne 0} \frac{\|\hat{f}_k\|}{|k\cdot\omega|} < \infty$. Note that $G(u) = \overline{f}(u, \omega, \theta_0)$, where

$$\overline{f}(x,\omega,\theta_0) := \lim_{T\to\infty} \frac{1}{T}\int_0^T f(x, \omega t + \theta_0)\, dt\,. \tag{3.17}$$

[DO WE WANT UNIFORM CONVERGENCE HERE?]

To summarize, we specialize to the systems

$$\frac{dx}{dt} = \varepsilon f(x, \omega t + \theta_0) \ =\ \varepsilon\, g(x,t), \tag{3.18}$$

$$\frac{du}{dt} = \varepsilon \overline{f}(u, \omega, \theta_0) \ =\ \varepsilon\, G(u), \tag{3.19}$$

$$\frac{dv}{dt} = \overline{f}(v, \omega, \theta_0) \ =\ G(v) \qquad \text{(guiding solution)}. \tag{3.20}$$

We then have

Corollary 1 *When the general KBM theorem is applied to the systems above (under additional assumptions to obtain the first inequality below), we have*

$$\delta(\varepsilon) = \varepsilon \sup_{(x,t)\in\overline{W}\times[0,T/\varepsilon]} \left\| \int_0^t \big(\widetilde{g}(x,s)\big)\, ds \right\| \tag{3.21}$$

$$\le\ 2\varepsilon \sum_{k\cdot\omega\ne 0} \|\hat{f}_k\| \frac{\sin((k\cdot\omega)t/2)}{|k\cdot\omega|} \ \le\ 2\varepsilon \sum_{k\cdot\omega \ne 0} \frac{\|\hat{f}_k\|}{|k\cdot\omega|} \ =\ 2\varepsilon\, C(f,\omega). \tag{3.22}$$

In the next section, we give concrete examples of how $C(f,\omega) < \infty$ may be realized for various ω.

4 Nonresonant, resonant, and Diophantine frequency vectors; application to KBM

In this section, we show how Corollary 1 (above) applies in the quasiperiodic case for various sets of frequency vectors. We first define and describe the sets of frequency vectors to be used.

4.1 Definitions and basic properties of various sets of frequency vectors

4.1.1 Diophantine and nonresonant frequency vectors

For positive parameters τ and γ, we define the sets

$$\mathcal{D}_{\tau,\gamma} = \{\omega \in \mathbb{R}^m \,|\, \text{for every } k \in \mathbb{Z}^m \backslash \{0\},\ |k \cdot \omega| \geq \gamma |k|^{-\tau}\}\,, \tag{4.23}$$

$$\mathcal{D}_{\tau} = \bigcup_{\gamma>0} \mathcal{D}_{\tau,\gamma}\,, \tag{4.24}$$

$$\mathcal{N} = \{\omega \in \mathbb{R}^m \,\big|\, k \in \mathbb{Z}^m \text{ and } k \cdot \omega = 0 \Rightarrow k = 0\}\,. \tag{4.25}$$

(Recall that $\mathcal{N}$ is the set of nonresonant frequency vectors.)

These sets have the following properties depending on the values of τ and γ:

(i) For all $\tau, \gamma > 0$, $\mathcal{D}_{\tau,\gamma} \subset \mathcal{D}_{\tau} \subset \mathcal{N} \subset \mathbb{R}^m$, and $\mathcal{N}$ is of full Lebesgue measure in $\mathbb{R}^m$.

(ii) For $0 < \tau < m-1$, $\mathcal{D}_{\tau,\gamma}$ is empty.

(iii) For $\tau \geq m-1$, and $\gamma > 0$ sufficiently small, $\mathcal{D}_{\tau,\gamma}$ is nonempty. Whenever $\mathcal{D}_{\tau,\gamma}$ is nonempty, it is Cantor-like (i.e., it is closed and has empty interior).

(iv) For $\tau > m-1$, the Lebesgue measure of $\overline{\mathcal{B}}_1 \backslash \mathcal{D}_{\tau,\gamma}$ is $O(\gamma)$ as $\gamma \to 0$. (Here $\overline{\mathcal{B}}_1$ is the closed unit ball centered on 0 in $\mathbb{R}^m$.)

(v) For $0 < \tau < m-1$, $\mathcal{D}_{\tau}$ is empty.

(vi) For $\tau = m-1$, $\mathcal{D}_{\tau}$ has Lebesgue measure zero and Hausdorff measure m.

(vii) For $\tau > m-1$, $\mathcal{D}_{\tau}$ is of full Lebesgue measure in $\mathcal{N}$ and hence also in $\mathbb{R}^m$.

For proofs of (most of) these facts, see Lochak and Meunier (Appendix 4, pp. 289–291, also referencing J.W.S. Cassels' book).

4.1.2 Resonant frequency vectors and their multiplicity

We recall that frequency vectors $\omega \notin \mathcal{N}$ are said to be resonant. Given a resonant frequency vector ω, there is a $k \in \mathbb{Z}^m \backslash \{0\}$ such that $k \cdot \omega = 0$. More generally, if $k^{(1)} \cdot \omega = 0, k^{(2)} \cdot \omega = 0, \ldots, k^{(p)} \cdot \omega = 0$ for precisely p linearly independent integer vectors $k^{(1)}, \ldots, k^{(p)}$, then we say that ω is resonant with multiplicity p (or ω is a p-multiple resonance, etc.). Clearly the multiplicity of $\omega = 0$ is m, and if $\omega \in \mathbb{R}^m \backslash \{0\}$ is resonant, its multiplicity p satisfies $1 \leq p \leq m-1$.

4.2 KBM results using various sets of frequency vectors

We now describe how Corollary 1 (§3.2) applies using the sets of frequency vectors just defined. We begin with $\mathcal{N}$, then move to smaller subsets on which the error estimates are progressively better. We also briefly describe results for resonant ω (i.e., frequency vectors in the complement of $\mathcal{N}$).

(a) Given any $\omega \in \mathcal{N}$, the Kronecker-Weyl theorem [REFERENCE OR STATEMENT?] combined with the KBM theorem shows that $\delta(\varepsilon) = o(1)$, but with the rate of decay of $\delta(\varepsilon)$ depending strongly—and very discontinously—on ω. [CAN WE SAY MORE HERE?]

(b) Fix $\tau > m - 1$. Given $\omega \in \mathcal{D}_\tau$, we define $\gamma(\omega) = \max\{\gamma \,|\, \omega \in \mathcal{D}_{\tau,\gamma}\}$. [CHECK THIS DEFINITION!]. Then for all $\omega \in \mathcal{D}_\tau$ and $k \in \mathbb{Z}^m \backslash \{0\}$, we have $|k \cdot \omega| \geq \gamma(\omega)|k|^{-\tau}$. Thus, for sufficiently smooth f and for any fixed $\omega \in \mathcal{D}_\tau$, $\delta(\varepsilon) \leq \frac{\varepsilon}{\gamma(\omega)} \sum_{k\neq 0} \|\hat{f}_k\| |k|^\tau < \infty$; in other words $\delta(\varepsilon) = \frac{1}{\gamma(\omega)} O(\varepsilon)$. Inequalities of this type hold on all of $\mathcal{D}_\tau$, and thus for almost all frequency vectors ω.

However, although this type of estimate is obviously better than the estimate in (a), nevertheless it is again strongly dependent on ω; in particular $\delta(\varepsilon)$ is not uniformly $O(\varepsilon)$ on $\mathcal{D}_\tau$ because $\inf_{\omega\in\mathcal{D}_\tau} \gamma(\omega) = 0$. [CHECK!]

(c) To obtain uniformity, a standard approach is to restrict ω to a subset of $\mathcal{D}_\tau$ such as $\mathcal{D}_{\tau,\gamma}$. More precisely, for fixed $\tau > m - 1$ and for sufficiently small (but fixed!) $\gamma > 0$, $\mathcal{D}_{\tau,\gamma}$ is nonempty. Given any $\omega \in \mathcal{D}_{\tau,\gamma}$, for sufficiently smooth f, we have $\delta(\varepsilon) \leq \frac{\varepsilon}{\gamma} \sum_{k\neq 0} \|\hat{f}_k\| |k|^\tau < \infty$ and thus $\delta(\varepsilon) = O(\varepsilon)$ uniformly on $\mathcal{D}_{\tau,\gamma}$. The error estimate for KBM is then $O(\varepsilon^{1/2})$ uniformly on $\mathcal{D}_{\tau,\gamma}$.

Although we do have uniformity in this case, we recall that the set $\mathcal{D}_{\tau,\gamma}$ is Cantor-like (see (iii), above) which makes its use in applications highly problematic.

** This may or may not be the place to say so, but we can again emphasize that our improvements consist of getting (1) a better, $O(\varepsilon)$, uniform error estimate on (2) a larger, nicer set of ω s (the cut-off Diophantine set). The cut-off Diophantine set is not Cantor-like, and is better for applications (I have some language for this in my QTDS paper).

(d) If ω is resonant of multiplicity p, then there is an $m \times m$ Smith normal form matrix S such that $S\omega = (0, \hat{\omega})^T$ where $0 \in \mathbb{R}^p$ and $\hat{\omega} \in \mathbb{R}^{m-p}$ with $\hat{\omega}$ nonresonant in $\mathbb{R}^{m-p}$. Thus the above procedure may be repeated in the lower dimensional frequency space $\mathbb{R}^{m-p}$.

(e) Finally, using the result of Siegel and Moser [described in Klaus and Jim's earlier paper??], there is a nonresonant ω, outside of all $\mathcal{D}_\tau$, such that $\delta(\varepsilon) = o(1)$ (and no better than this), which shows that the KBM estimate is sharp over all $\omega \in \mathcal{N}$.

5 Improved averaging method for frequency vectors far from low-order resonance (FLOR)

5.1 Motivation

If $\omega \in \mathcal{N}$, the Kronecker-Weyl theorem ensures that the KBM average exists and equals $\hat{f}_0$. This can be turned around to ask: For what ω is $\hat{f}_0$ a good normal form?

As described above, the standard answers are to take $\omega \in \mathcal{D}_\tau$, giving $\delta(\varepsilon) \leq C(\omega)\varepsilon$; or take $\omega \in \mathcal{D}_{\tau,\omega}$, giving $\delta(\varepsilon) \leq C\varepsilon$ uniformly over $\mathcal{D}_{\tau,\omega}$. While both of these have a certain simplicity and elegance from the mathematical viewpoint, as discussed above, they also have shortcomings when used in applications. We address these shortcomings with our improved method below.

5.2 Description of our method

Here we pick $\varepsilon > 0$ and Diophantine parameters $\tau > m-1$ and $\gamma > 0$ in advance, generating the nonempty cut-off Diophantine set $\mathcal{D}_{\tau,\gamma}\big(N(\varepsilon)\big)$ and leading to the estimate $\Delta(\varepsilon) = O(\varepsilon/\gamma)$ which is valid uniformly over $\omega \in \mathcal{D}_{\tau,\gamma}\big(N(\varepsilon)\big)$. Thus $\hat{f}_0$ is a good normal form for these ω, and we emphasize that $\mathcal{D}_{\tau,\gamma}\big(N(\varepsilon)\big)$ is a nice set with finitely many connected components and nonempty interior (not a Cantor-like set).

——[INSERT PASTED FROM DEH42 BEGINS HERE]——

Here we consider the case where ω in (1.3) is far from low-order resonance (FLOR) as defined in terms of the cutoff Diophantine set $\mathcal{D}_{\tau,\gamma}(R)$ defined below. In our FLOR case the appropriate approximate solution u is given by the initial value problem

$$\frac{du}{dt} = \varepsilon \hat{f}_0(u) \ , \quad u(0) = \xi \ , \tag{5.26}$$

where $\hat{f}_0 = \hat{f}_0(x)$ is the zeroth Fourier coefficient of f in Eq. (1.1). This is the KBM normal form when ω is nonresonant.

We now present a detailed set-up for using the solution $u = u(t,\varepsilon)$ of (5.26) to approximate the solution $x = x(t,\varepsilon)$ of (1.3). We begin by constructing a tubular domain W around the guiding solution.

5.3 The domain W

We begin by defining the open set $W \subset U$ with compact closure which will be our working domain. [CHECK THAT THE ORDER OF CHOOSING THINGS IS CORRECT IN THE SET-UP LIST.] Consider the guiding solution v defined by $v(t) := u(t/\varepsilon)$ so that

$$\frac{dv}{dt} = \hat{f}_0(v) \ , \quad v(0) = \xi \ . \tag{5.27}$$

In Appendix [#] it is shown that every Fourier coefficient $\hat{f}_k$ of f in Eq. (1.3) is of class C^1. This is a sufficient condition for the standard theorems on local existence and uniqueness and extension to a maximal existence interval to hold (see e.g. [Hale?]), and so we can apply them to our initial value problems (1.3), (5.26), and (5.27).

Now pick $\xi \in U$ and let $[0, T_g(\xi))$ denote the maximal forward existence interval in U for the guiding solution v. Next choose $T \in (0, T_g(\xi))$ in which case

$$S(\xi, T) := \{v(t) : 0 \le t \le T\} \tag{5.28}$$

is a compact subset of U. Since $S(\xi,T)$ is a compact subset of the open set U, the distance between $S(\xi,T)$ and $\mathbb{R}^d \setminus U$, defined by

$$\mathrm{dist}(S(\xi,T), \mathbb{R}^d \setminus U) := \begin{cases} \inf_{\substack{x \in (\mathbb{R}^d \setminus U) \\ s \in S(\xi,T)}} & \text{if } U \neq \mathbb{R}^d \\ +\infty & \text{if } U = \mathbb{R}^d \end{cases} \ , \tag{5.29}$$

is positive. We choose the positive parameter $\delta < \text{dist}(S(\xi,T), \mathbb{R}^d \setminus U)$, and we define W as the set of points within a distance δ of $S(\xi,T)$, more precisely:

$$W = W(\xi,T) := \bigcup_{s \in S(\xi,T)} B_\delta(s) \,, \tag{5.30}$$

where $B_\delta(s)$ is the open ball of radius δ centered at s. The compactness of $S(\xi,T)$ guarantees that the closure of W is given by

$$\overline{W} = \bigcup_{s \in S(\xi,T)} \overline{B_\delta(s)} \,, \tag{5.31}$$

where $\overline{B_\delta(s)}$ is the closed ball of radius δ centered at s. It follows that $\overline{W}$ is a compact subset of U.

Finally, we take $[0, \beta(\varepsilon))$ to be the maximal forward existence interval for the IVP (1.3) when x is restricted to W.

5.4 The cut-off Diophantine set $\mathcal{D}_{\tau,\gamma}(N)$

We now define what we mean by "FLOR" by precisely defining the set of admissible ω so that (5.26) is a good approximation to (1.3). For $\tau > m - 1$ and $\gamma > 0$ we define the standard "zone function" $\phi_{\tau,\gamma} : (0,\infty) \to (0,\infty)$ for $r \in (0,\infty)$ by

$$\phi_{\tau,\gamma}(r) := \gamma r^{-\tau} \,, \tag{5.32}$$

and the "cut-off Diophantine set" $\mathcal{D}_{\tau,\gamma}(R)$ by

$$\mathcal{D}_{\tau,\gamma}(R) := \bigcap_{\substack{k \in \mathbb{Z}^m \\ 0 < ||k|| \le R}} \{\Omega \in \mathbb{R}^m : |k \cdot \Omega| \ge \phi_{\tau,\gamma}(||k||)\} \,, \tag{5.33}$$

where $R \in [0,\infty)$. Using $\alpha \ge m + 2$, it follows from Appendix [#] of [ ] that $N = N(\varepsilon, \delta, \xi, T)$, defined by

$$N = N(\varepsilon, \xi, T) := min\{N' \in \{0,1,2\ldots\} : \sum_{\substack{k \in \mathbb{Z}^m \\ ||k|| > N'}} \sup_{y \in \overline{W}} (|\hat{f}_k(y)|) \le \delta\varepsilon\} \,, \tag{5.34}$$

is finite and thus

$$\sum_{\substack{k \in \mathbb{Z}^m \\ ||k|| > N}} \sup_{y \in \overline{W}} (|\hat{f}_k(y)|) \le \delta\varepsilon \,. \tag{5.35}$$

where $\delta \in (0,\infty)$.

The "cut-off Diophantine set" $\mathcal{D}_{\tau,\gamma}(N)$ given by (5.33) is our set of admissible ω for the FLOR case. We discuss the size of this set later.

5.5 Statement of Theorem 1

By restricting x (solution of the exact IVP ( )) to W we may now formulate:

Theorem 1 *Let d, m, α be positive integers with $\alpha \geq m+2$, and let ξ be an initial condition in the open domain $U \subset \mathbb{R}^d$. Assume the vector field $f \in \mathfrak{C}^{\alpha}_{per,m}(U, \mathbb{R}^d)$. Choose the following positive parameters: the perturbation strength ε, zone parameters γ and $\tau < \alpha - m - 1$, and a time constant $T < T_g$, where $T_g = T_g(\xi)$ is the maximal forward existence time in U of the initial value problem (5.27) (the guiding problem). Define the tubular set $W = W(\xi, T)$ around the guiding solution by (5.30) and the cutoff $N = N(\varepsilon, \xi, T)$ by (5.34). Assume that the frequency vector ω is far from low-order resonance, in other words $\omega \in \mathcal{D}_{\tau,\gamma}(N)$, where the cut-off Diophantine set $\mathcal{D}_{\tau,\gamma}(N)$ is defined by (5.33). Finally, let $\beta(\varepsilon)$ denote the maximal forward existence time in W of the solution $x = x(t)$ of the (exact) initial value problem (1.3), and let $u = u(t)$ be the solution of the (averaged) initial value problem (5.26) in W (u is the approximation to x). Then there exists a positive constant $K = K(f, \xi, \tau, \gamma, T)$ such that for all $t \in [0, T/\varepsilon] \cap [0, \beta(\varepsilon))$, the error $|x - u|$ between the exact and approximate solutions satisfies the estimate*

$$|x(t) - u(t)| \leq \frac{\varepsilon}{\gamma} K \ . \tag{5.36}$$

We now make some remarks on Theorem 1. First, the condition $m - 1 < \tau < \alpha - m - 1$ implies $\alpha \geq 2m + 1$ (since α is an integer). Second, because of ( ), in the case $m = 1$, the cut-off Diophantine set $\mathcal{D}_{\tau,\gamma}(N)$ has the remarkably simple form

$$\mathcal{D}_{\tau,\gamma}(N) = \begin{cases} \mathbb{R} \setminus (-\gamma, \gamma) & \text{if } N \geq 1 \ , \\ \mathbb{R} & \text{otherwise} \ . \end{cases} \tag{5.37}$$

Thus, when $m = 1$ and $N \geq 1$, all ω not too close to zero are allowed. Finally, an explicit form of the order constant K is given below in ( ).

5.6 Proof of Theorem 1

Assume the hypotheses of the theorem.

We will use the BCL with $g(x,t) = f(x,t)$ and $G(u,s) = \hat{f}_0(x)$ (the 0th Fourier coefficient of f). This means that we need to show that $B(t,\varepsilon) = |\int_0^t \tilde{f}(v(\varepsilon s), s)\, ds\,|$ is bounded for $t \in [0, T/\varepsilon]$, where we define $\tilde{f}$, the "fluctuating part" of f, as $\tilde{f}(v,s) = f(v,s) - \hat{f}_0(v)$.

We begin by decomposing $\tilde{f}$ into a truncated part

$$\tilde{f}_{\leq}(x,t) := \sum_{\substack{0 \neq k \in \mathbb{Z}^m \\ ||k|| \leq N(\varepsilon)}} \hat{f}_k(x) \exp(i(k \cdot \omega)t) \ , \tag{5.38}$$

and a tail portion

$$\tilde{f}_{>}(x,t) := \tilde{f}(x,t) - \tilde{f}_{\leq}(x,t) \sim \sum_{\substack{0 \neq k \in \mathbb{Z}^m \\ ||k|| > N(\varepsilon)}} \hat{f}_k(x) \exp(i(k \cdot \omega)t) \ . \tag{5.39}$$

We first deal with the tail portion.

By taking $\tilde{f}_k(x,\theta) := \hat{f}_k(x)\, \exp(i(k \cdot \theta))$, we show in Appendix [#] that the family $(\tilde{f}_k(x,\cdot))_{k \in \mathbb{Z}^m}$ is absolutely summable in the Banach space $(\mathfrak{B}(\mathbb{R}^m, \mathbb{C}^d), |\cdot|_{sup})$ of functions from $\mathbb{R}^m$ to $\mathbb{C}^d$ which are bounded with respect to the Euclidean norm $|\cdot|$ on $\mathbb{C}^d$. It follows that the subfamily $(\tilde{f}_k(x,\cdot))_{\substack{k \in \mathbb{Z}^m \\ ||k|| > N(\varepsilon)}}$ is also absolutely summable in $(\mathfrak{B}(\mathbb{R}^m, \mathbb{C}^d), |\cdot|_{sup})$. Thus for $x \in W, t \in \mathbb{R}$, we have the following simple estimate for the tail $\tilde{f}_{>}$:

$$|\tilde{f}_{>}(x,t)| \leq \sum_{\substack{k \in \mathbb{Z}^m \\ ||k|| > N(\varepsilon)}} |\hat{f}_k(x)| \leq \sum_{\substack{k \in \mathbb{Z}^m \\ ||k|| > N(\varepsilon)}} \sup_{x \in \overline{W}} (|\hat{f}_k(x)|) \leq \delta\varepsilon \ , \tag{5.40}$$

where the last inequality follows from ( ).

Next, we estimate the integral of the truncated part of $\tilde{f}$ as

$$\left| \int_0^t \tilde{f}_{\leq}(u(s), s) ds \right| \leq \sum_{\substack{0 \neq k \in \mathbb{Z}^m \\ ||k|| \leq N(\varepsilon)}} \left| \int_0^t \hat{f}_k(u(s)) \exp(i(k \cdot \omega)s) ds \right| . \tag{5.41}$$

An integration by parts inside the summand gives

$$\begin{aligned} &\left| \int_0^t \hat{f}_k(u(s)) \exp(i(k \cdot \omega)s) ds \right| \\ &= \left| \frac{1}{ik \cdot \omega} \left(\hat{f}_k(u(t)) \exp(i(k \cdot \omega)t) - \hat{f}_k(u(0)) - \varepsilon \int_0^t D\hat{f}_k(u(s)) \hat{f}_0(u(s)) \exp(ik \cdot \omega s) ds \right) \right| \\ &\leq \frac{1}{|k \cdot \omega|} \left(2 \sup_{x \in \overline{W}} (|\hat{f}_k(x)|) + T \sup_{x \in \overline{W}} (|D\hat{f}_k(x)|_{ind}) M_0 \right) , \end{aligned} \tag{5.42}$$

where

$$M_0 := \sup_{x \in \overline{W}} (|\hat{f}_0(x)|) . \tag{5.43}$$

and we have used $\varepsilon t \leq T$. It follows that

$$\left| \int_0^t \tilde{f}_{\leq}(u(s), s) ds \right| \leq \sum_{\substack{0 \neq k \in \mathbb{Z}^m \\ ||k|| \leq N(\varepsilon)}} \frac{1}{|k \cdot \omega|} \left(2 \sup_{x \in \overline{W}} (|\hat{f}_k(x)|) + T \sup_{x \in \overline{W}} (|D\hat{f}_k(x)|_{ind}) M_0 \right) . \tag{5.44}$$

Applying the Diophantine condition, $\frac{1}{|k \cdot \omega|} \leq \frac{||k||^\tau}{\gamma}$, for $0 \neq ||k|| \leq N(\varepsilon)$, and then summing over all k gives

$$\left| \int_0^t \tilde{f}_{\leq}(u(s), s) ds \right| \leq \frac{1}{\gamma} (2M_1 + TM_2M_0) , \tag{5.45}$$

where

$$M_1 := \sum_{k \in \mathbb{Z}^m} ||k||^\tau \sup_{x \in \overline{W}} (|\hat{f}_k(x)|) , \quad M_2 := \sum_{k \in \mathbb{Z}^m} ||k||^\tau \sup_{x \in \overline{W}} (|D\hat{f}_k(x)|_{ind}) . \tag{5.46}$$

Since $0 < \tau < \alpha - m - 1$ we have, by ( ) and ( ) that M_1 and M_2 are finite. In fact

$$M_1 \leq C_1(\alpha, m) \Upsilon(\alpha - \tau, m) , \quad M_2 \leq C_2(\alpha, m, d) \Upsilon(\alpha - \tau - 1, m) , \tag{5.47}$$

where

$$\Upsilon(r, m) := \sum_{0 \neq k \in \mathbb{Z}^m} ||k||^{-r} , \qquad (r > m) . \tag{5.48}$$

We are now in position to get a bounding function $\widehat{B}(t, \varepsilon)$ as required by the BCL. We have $B(t, \varepsilon) = |\int_0^t \tilde{f}(v(\varepsilon s), s) \, ds| \leq |\int_0^t \tilde{f}_{\leq}(v(\varepsilon s), s) \, ds| + \int_0^t |\tilde{f}_{>}(v(\varepsilon s), s)| \, ds \leq \frac{1}{\gamma}(2M_1 + TM_2M_0) + \delta \varepsilon t = \widehat{B}(t, \varepsilon)$.

Thus, applying the BCL gives

$$|x(t) - u(t)| \ \leq \frac{\varepsilon}{\gamma}\Big(2M_1 + T(\delta\gamma + M_2 M_0)\Big)e^{LT} \ . \tag{5.49}$$

By taking $\delta = 1/\gamma$ (which is good since it gives a smaller $N(\varepsilon)$) we obtain

$$|x(t) - u(t)| \leq \frac{\varepsilon}{\gamma}K_1(T) \ , \tag{5.50}$$

where

$$K_1(T) := \Big(2M_1 + T(1 + M_2 M_0)\Big)e^{LT} \ . \tag{5.51}$$

——[INSERT PASTED FROM DEH42 ENDS HERE]——

6 Averaging at or near resonance

6.1 Motivation

If ω is resonant, KBM shows that the KBM average exists and equals $\sum_{k\in\omega^\perp} \hat{f}_{(}x)e^{ik\cdot\theta}$, which is an average over a "subtorus" (as discussed—at least obliquely—in Lochak and Meunier: §7.4; Appen. 3, §8; & Appen. 8, §2). Again, we turn this around to ask: For what ω is this average a good normal form? Our answer: It's good for $\omega = \omega_r + \varepsilon a$, where ω_r is "sufficiently resonant."

6.2 Description of our method

6.2.1 At resonance

Choose a resonant frequency vector ω_r (multiplicity p), leading in turn to a corresponding Smith pair (S, λ). Assume that λ is Diophantine in the appropriate way. Then we get an estimate of the form $\Delta(\varepsilon) = O(\varepsilon\|S^{-T}\|/\gamma(\lambda))$.

6.2.2 Near low-order resonance (NLOR)

Here again we choose Diophantine parameters $\gamma > 0$ and $\tau > m - 1$ in advance, generating the appropriate nonempty Diophantine set, and leading to the estimate $\Delta(\varepsilon) = O(\varepsilon\|S^{-T}\|/\gamma)$, again valid uniformly over the Diophantine set.

LEFTOVER FROM PREVIOUS DISCUSSION OF KBM:

Now define the oscillating part of f by $\widetilde{f}(x,t) = f(x, \omega t + \beta) - \overline{f}(x, \omega, \beta)$, and let

$$\delta(\varepsilon) := \varepsilon \sup_{(x,t)\in K_0\times[0,T/\varepsilon]} \Big\| \int_0^t \widetilde{f}(x,s)\, ds \Big\| . \tag{6.52}$$

Then Sáenz' lemmas 1 and 2 lead to

1. $|x(t,\varepsilon) - v(\varepsilon t)| \leq C\sqrt{\delta(\varepsilon)}\, e^{LT}$ for $t \in [0, T/\varepsilon]$ and
2. $\delta(\varepsilon) = o(1)$ as $\varepsilon \to 0$ for all ω.

Next, we could get interesting bounds on $\delta(\varepsilon)$ in the QP case by following Jim's 8/15/12 notes. This leads (for $\beta = 0$ and ω NR) to the bound

$$\sup_{t\in[0,T/\varepsilon]} \left\| \int_0^t \widetilde{f}(x,s)\,ds \right\| \;\le\; 2 \sum_{k\neq 0} |\hat{f}_k(x)| \sup_{t\in[0,T/\varepsilon]} \frac{|\sin(k\cdot\omega)t/2|}{|k\cdot\omega|} \tag{6.53}$$

QUESTIONS

1. In the KBM context, what is different about the cases ω resonant/nonresonant (besides the normal forms)?

2. Does $\beta = \varepsilon a t$ fit into KBM in an interesting way?

Since, for $k\cdot\omega \neq 0$, we have $\frac{1}{T}\int_0^T \exp i(k\cdot\omega)t \exp i(k\cdot\beta)\,dt = \frac{e^{i(k\cdot\omega)T}-1}{iTk\cdot\omega} e^{ik\cdot\beta} \to 0$ as $T\to\infty$, it follows that $\overline{f}(x,\omega,\beta) = \sum_{k\in\omega^\perp} \hat{f}_k(x) e^{ik\cdot\beta}$, where $\omega^\perp = \{k \in \mathbb{Z}^m \mid k\cdot\omega = 0\}$.

If ω is nonresonant ($\omega^\perp = \{0\}$), the K-W lemma ensures that $\overline{f}(x,\omega,\beta) = \hat{f}_0(x)$.

If ω is resonant ($\omega\cdot k = 0$ for some $k \neq 0$), then there is a matrix $S \in SL_m(\mathbb{Z})$ such that $S\omega = (0,\lambda)$, and the K-W lemma can be used to show that $\overline{f}(x,\omega,\beta)$ is an average over a subtorus.