# DETAILS OF ORBITAL EIGEN-ANALYSIS FOR ELECTRON STORAGE RINGS

J. ELLISON, H. MAIS, K. HEINEMANN, G. RIPKEN (DECEASED 2004)



## 1. Introduction

This paper is an expansion of the article that appears in the Handbook article of [1]. Dedicate to the memory of Ripken?

## 2. Basic problem

A general 6-D formalism is presented for the calculation of the bunch parameters (e.g. 6-D stationary beam-envelope matrix) for electron storage rings including radiation damping and quantum excitation. The problem is formulated in terms of a stochastic differential equation (SDE) and basic to our approach is the orbital eigen-analysis first introduced in [2]. The latter gives a more general framework than that of Courant-Synder. An SDE approach was first introduced in [3] and developed further in [4]. At a later stage work [5] was incorporated and the starting point is the SDE in Frenet-Serret coordinates with respect to a design orbit as described in [6]. The 6D-SDE is

$$\vec{x}' = \mathbf{A}(s)\vec{x} + \vec{c}(s) + \epsilon\vec{g}(\vec{x}, s) + \epsilon^{\frac{1}{2}}\sqrt{b(s)}\xi(s)\vec{e}_6, \tag{1}$$

where $\vec{x} = (x, p_x, y, p_y, z, p_z)^t$. We have expanded up to second order in the dynamical variables and retained only the leading nonlinearities due to sextupoles and due to radiation effects in quadrupoles. All functions except $\xi$ are $C$-periodic in $s$ where $C$ is the ring circumference. The matrix $\mathbf{A}(s)$ is Hamiltonian and gives rise to the linear symplectic synchro-betatron oscillations. The $\vec{x}$-independent term $\vec{c}$ describes closed orbit distortions induced by dipole field errors and by the fact that the energy losses in the bending magnets and quadrupoles are not replaced at the location in the ring where they occur. Our analysis is perturbative. The parameter $\epsilon$ is inserted to indicate the perturbation size and to discuss the perturbation procedure and the nature of the error estimates; the $\epsilon^{1/2}$ in the noise gives the right balance so that the two perturbation terms enter the solution at the same order. In the end, $\epsilon$ can be taken to be one in applications of the formulas. The first perturbation term is $\vec{g}(\vec{x}, s) = \delta\mathbf{A}(s)\vec{x} + \vec{f}(\vec{x}, s)$. Here $\delta\mathbf{A}$ models both the energy losses from radiation and the energy gain in the rf cavities, and $\vec{f}$ takes into account the nonlinear terms due to sextupoles and due to radiation effects in quadrupoles. The last term in (1) simulates the stochastic excitation of the particle motion due to the quantum nature of the radiation. Here $\xi$ is Gaussian white noise, $b$ is an amplitude function proportional to $\hbar$ and $\vec{e}_k$ is the unit vector with 1 in the $k$-component, thus the stochastic excitation only

affects the $p_z$ component directly. The explicit form of these quantities can be found in [6] and an outline of the derivation is given in the Appendix.

THINGS TO ADD:

(1) Discussion of $\epsilon$ and $\epsilon^{1/2}$

## 3. Random Bunch Density

The main quantity of interest is the $N$ particle random bunch density

$$\rho_N(\vec{x}, s) := \frac{1}{N}\sum_{n=1}^{N} \delta(\vec{x} - \vec{x}_n(s)), \tag{2}$$

where the $\vec{x}_n(s)$ are independent and identically distributed random variables determined by (1). Let $p$ be the single particle probability density defined by (1), then $\langle \rho_N \rangle = p$. Here, and in the following, angular brackets will denote the expected value of stochastic quantities. We will assume for large $N$ that $\rho_N(\vec{x}, s) \approx p(\vec{x}, s)$, in a coarse grained sense. This article presents results of an analysis of this single particle probability density. More precisely, the (periodic) 6-D closed orbit $\vec{x}_{co}$, satisfying (1) with $b = 0$ will be defined by an integral equation, (1) will be linearized around $\vec{x}_{co}$ and the linearized equation analyzed.

Things to Add:

(1) Coarse graining, CLT and Law of Large Numbers

## 4. The equation $\vec{x}' = \mathbf{A}(s)\vec{x}$ and its Eigen-FSM

The solutions of the linear periodic Hamiltonian system

$$\vec{x}\,' = \mathbf{A}(s)\vec{x}, \mathbf{A}^t\mathbf{J} + \mathbf{J}\mathbf{A} = 0, \mathbf{A}(s+C) = \mathbf{A}(s) \tag{3}$$

are central to our analysis. Here $\mathbf{J} = \mathrm{diag}(\mathbf{J}_2, \mathbf{J}_2, \mathbf{J}_2)$ is the unit symplectic matrix where $\mathbf{J}_2 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$.

The transfer map, $\mathbf{M}(s, s_0)$, is a fundamental solution matrix (FSM) with independent variable $s$ which satisfies $\mathbf{M}(s_0, s_0) = I$ (often called the principal solution matrix). The basic properties are

**Transport Property:** $\mathbf{M}(s_2, s_1)\mathbf{M}(s_1, s_0) = \mathbf{M}(s_2, s_0)$
**Symplecticity:** $\mathbf{M}^t\mathbf{J}\mathbf{M} = \mathbf{J}$
**Periodicity:** $\mathbf{M}(s + C, s_0 + C) = \mathbf{M}(s, s_0)$

We assume solutions of (3) are bounded. This is the case if and only if the monodromy matrix, $\mathbf{M}(C, 0)$, has six linearly independent eigenvectors, $\vec{w}_k$, and its eigenvalues (*characteristic multipliers*), $\rho_k = \exp(\imath 2\pi\nu_k)$, have modulus one where the $\nu_k$ are tunes (*characteristic exponents*), see [7]. To avoid resonance, we further assume that the $\rho_k$ are distinct. Because $M$ is real we can choose the $\nu_k$ such that

$$0 < \nu_1 < \nu_3 < \nu_5 < 1/2, \;\; \nu_{2l} = -\nu_{2l-1}, \tag{4}$$

and the $\vec{w}_k$ such that $\vec{w}_{2l} = \vec{w}^*_{2l-1}$. Thus $(1-\rho_j^*\rho_k)\vec{w}_j^H\mathbf{J}\vec{w}_k = 0$, so that $\vec{w}_j^H\mathbf{J}\vec{w}_k = 0$ for $j \neq k$. Let $\vec{w}_k = \vec{a}_k + i\vec{b}_k$, $\vec{a}_k$ and $\vec{b}_k$ real then $\vec{w}_k^H\mathbf{J}\vec{w}_k = i\gamma_k$ where $\gamma_k = 2\vec{a}_k^t\mathbf{J}\vec{b}_k$ and $\gamma_{2l} = -\gamma_{2l-1}$. More compactly

$$\mathbf{W}^H\mathbf{J}\mathbf{W} = i\tilde{\mathbf{I}}, \text{ where} \tag{5}$$

$$\mathbf{W} = [\vec{w}_1, \cdots, \vec{w}_6],\ \tilde{\mathbf{I}} = \mathrm{diag}(\gamma_1, \cdots, \gamma_6).$$

Finally, the $\gamma_k$ are nonzero (since $\mathbf{W}$ is nonsingular) and we normalize the $\vec{w}_{2l-1}$ so that $\gamma_{2l-1} = \pm 1$ (it appears the sign is not known a priori).

The phase space density of a bunch can be efficiently approximated in terms of the Eigen-FSM for (3),

$$\mathbf{\Psi}(s) := \mathbf{M}(s,0)\mathbf{W}, \tag{6}$$

which was first introduced in [2] and generalizes the Courant-Synder formalism. It follows from (5) and (P2) that

$$\mathbf{\Psi}^H(s)\mathbf{J}\mathbf{\Psi}(s) = i\tilde{\mathbf{I}}. \tag{7}$$

Define $\hat{\mathbf{\Psi}}$ by

$$\mathbf{\Psi}(s) =: \hat{\mathbf{\Psi}}(s)\exp(\imath\omega s\mathbf{N}),\ \omega := 2\pi/C, \tag{8}$$

where $\mathbf{N} = \mathrm{diag}(\nu_1, \cdots, \nu_6)$, then $\hat{\mathbf{\Psi}}$ is $C$-periodic and (8) is a Floquet representation of the Eigen-FSM. Furthermore $\hat{\mathbf{\Psi}}$ satisfies (7). The $k$-th column of $\hat{\mathbf{\Psi}}$ will be denoted by $\vec{\hat{\psi}}_k$.

THING TO ADD:

(1) Proof of bounded iff modulus 1 and full set of e-vectors.
(2) some discussion of resonance
(3) Two different J's
(4) J orthogonality when full set of eigenvectors but eigenvalues not distinct.

## 5. Closed orbit and associated linearized SDE

Since no characteristic multiplier of (3) is one, Eq.(1), with $b = 0$, has a unique $C$-periodic solution for $\epsilon$ sufficiently small. This closed orbit solution is defined by the integral equation

$$\begin{aligned}\vec{x}_{co}(s,\epsilon) &= \vec{x}_{co}(s,0)\\ &+\epsilon\textstyle\int_0^C \mathbf{G}(s,t)\vec{g}(\vec{x}_{co}(s+t,\epsilon), s+t)dt.\end{aligned} \tag{9}$$

Here $\mathbf{G}(s,t) = (\mathbf{M}(s,s+C) - \mathbf{I})^{-1}\mathbf{M}(s,s+t)$ and $\vec{x}_{co}(s,0) = \int_0^C \mathbf{G}(s,t)\vec{c}(s+t)dt$. See Thm. 2.1, p.154 of [8]. The closed orbit can be determined approximately by iterating (9).

Let $\vec{y} := \vec{x} - \vec{x}_{co}$ then for $\vec{y}$ small, $\vec{y} \approx \vec{y}_L$ where $\vec{y}_L$ satisfies the linear SDE

$$\vec{y}_L{}' = (\mathbf{A}(s) + \epsilon\mathbf{B}(s))\vec{y}_L + \epsilon^{\frac{1}{2}}\sqrt{b(s)}\xi(s)\vec{e}_6 \tag{10}$$

Here $\mathbf{B}(s) := D_1\vec{g}(\vec{x}_{co}(s), s)$ is the Jacobian matrix of $\vec{g}(\cdot, s)$. The most important information about the bunch is contained in the moment (beam-envelope) matrix $\mathbf{U}(s) =$

$\langle \vec{y}_L(s)\vec{y}_L^t(s)\rangle$. We now determine an approximation to $\mathbf{U}$.

THINGS TO ADD:

(1) Derivation of integral equation
(2) Outline of proof
(3) Discussion of iteration, e.g., one Des uses.

## 6. Equation for moment matrix and averaging approximation

The transformation $\vec{y}_L \to \vec{z}$ via $\vec{y}_L =: \mathbf{\Psi}(s)\vec{z}$ gives

$$\begin{aligned} \vec{z}\,' &= \epsilon \mathbf{D}(s)\vec{z} + \epsilon^{\frac{1}{2}}\xi(s)\vec{d}(s), \\ \vec{d}(s) &= \sqrt{b(s)}\mathbf{\Psi}^{-1}(s)\vec{e}_6, \\ \mathbf{D}(s) &:= \mathbf{\Psi}^{-1}(s)\mathbf{B}(s)\mathbf{\Psi}(s). \end{aligned} \tag{11}$$

Now $\mathbf{U} = \mathbf{\Psi}\mathbf{V}\mathbf{\Psi}^H$ where $\mathbf{V} = \langle \vec{z}\,\vec{z}^H\rangle$ and the differential equation for $\mathbf{V}$ is

$$\mathbf{V}' = \epsilon[\mathbf{D}(s)\mathbf{V} + \mathbf{V}\mathbf{D}^H(s) + \mathbf{E}(s)], \tag{12}$$

where $\mathbf{E}(s) = \vec{d}(s)\vec{d}^H(s)$.

Applying averaging methodology [9] to (12) we obtain $\mathbf{V}(s) \approx \mathbf{V}_a(s)$ where

$$\mathbf{V}_a' = \epsilon[\bar{\mathbf{D}}\mathbf{V}_a + \mathbf{V}_a\bar{\mathbf{D}}^H + \bar{\mathbf{E}}]. \tag{13}$$

Here $\bar{\mathbf{D}}$ and $\bar{\mathbf{E}}$ denote the average of the quasiperiodic functions $\mathbf{D}(s)$ and $\mathbf{E}(s)$, e.g.,

$$\bar{\mathbf{D}} = \lim_{L\to\infty}\frac{1}{L}\int_0^L \mathbf{D}(s)ds.$$

Because of the non-resonant tune condition (4), the averages $\bar{\mathbf{D}}$ and $\bar{\mathbf{E}}$ are diagonal with

$$\bar{\mathbf{D}}_{jj} = \lambda_j := -\imath\gamma_j\overline{\vec{\hat{\psi}}_j^H(s)\mathbf{J}\mathbf{B}(s)\vec{\hat{\psi}}_j(s)}, \tag{14}$$

$$\bar{\mathbf{E}}_{jj} = \overline{b(s)|\hat{\mathbf{\Psi}}(s)_{5j}|^2}, \tag{15}$$

where the overbar denotes the $s$ average of the underlying periodic function.

The solution of the IVP for (13) is

$$\begin{aligned} \mathbf{V}_a(s)_{jk} &= \exp[\epsilon(\lambda_j + \lambda_k^*)s](\mathbf{V}_0 - \mathbf{V}_{ae})_{jk} \\ &\quad + \mathbf{V}_{ae\,jk}, \\ \mathbf{V}_{ae\,jk} &= -\delta_{jk}\bar{\mathbf{E}}_{jj}/2\Re\lambda_j. \end{aligned} \tag{16}$$

Here $\mathbf{V}_0$ is the initial condition for (12) and $\mathbf{V}_{ae}$ is the unique equilibrium solution of (13).

In summary, the stochastic process $\vec{x}$, defined by (1), is given in a linear approximation by $\vec{x}_{co} + \vec{y}_L$ and the moment matrix of $\vec{y}_L$ is $\mathbf{U} = \mathbf{\Psi}\mathbf{V}\mathbf{\Psi}^H$, where the averaging approximation to the moment matrix is

$$\mathbf{U}(s) \approx \mathbf{U}_a(s) := \mathbf{\Psi}(s)\mathbf{V}_a(s)\mathbf{\Psi}^H(s). \tag{17}$$

THINGS TO ADD:

(1) Put in the $D_{ij}$ so the resonance effect becomes obvious; also one will see this in the averaging proof. Since the proof is quite simple in this case I propose putting it in, at least in an appendix.
(2) Apply Friedlin and Wentzel
(3) Does Sanders et al have the QP averaging result; if not I need to change the averaging reference in the HB article.
(4) Could it be that bounded + full set of eigenvectors still yields a J orthogonal set, but that the average of D is no longer diagonal.
(5) We note that the solution of (12) can be written in terms of the PSM for D as follows. Let $\Phi(s)$ be the PSM, i.e., $\Phi' = \mathbf{D}(s)\Phi,\ \ \Phi(0) = 0$, then

$$\mathbf{V}(s) = \Phi(s)\big[\mathbf{V}(0) + \int_0^s \Phi^{-1}(s')\mathbf{E}(s')\Phi^{-H}(s')ds'\big]\Phi^H(s)$$

## 7. Dissipative case

For $\Re\lambda_j < 0$, the dissipative case, $\mathbf{V}_a(s) \to \mathbf{V}_{ae}$ as $s \to \infty$ and the approximate moment matrix in (17) becomes $\mathbf{U}_a(s) = \mathbf{\Psi}(s)\mathbf{V}_{ae}\mathbf{\Psi}^H(s) = \hat{\mathbf{\Psi}}(s)\mathbf{V}_{ae}\hat{\mathbf{\Psi}}^H(s)$. Using (14), (15) and (16), this stationary, $C$-periodic $\mathbf{U}_a$ can be written

$$\begin{aligned}\mathbf{U}_a(s)_{jk} &= \sum_{l=1,2,3} G_l \Re[(\vec{\hat{\psi}}_{2l-1}(s))_j(\vec{\hat{\psi}}^*_{2l-1}(s))_k] \\ G_l &= -\frac{2}{\alpha_l}\frac{1}{C}\int_0^C b(s)|\vec{\hat{\psi}}_{2l-1}(s)_5)|^2 ds \end{aligned} \tag{18}$$

The quantities $\alpha_l := 2\Re\lambda_{2l-1}$ are called the damping constants and are given by

$$\begin{aligned}\Re\lambda_{2l-1} &= \Re\lambda_{2l} \\ &= \frac{\gamma_{2l-1}}{C}\int_0^C \Im[\vec{\hat{\psi}}_j^H(s)\mathbf{JB}(s)\vec{\hat{\psi}}_j(s)]ds < 0\end{aligned}$$

f. Robinson Sum Rule. Since $\mathbf{D}$ and $\mathbf{B}$ are related by the similarity transformation in (11), $\mathrm{Tr}\mathbf{D}(s) = \mathrm{Tr}\mathbf{B}(s)$ and thus the $C$-periodicity of $\mathbf{B}$ gives

$$\mathrm{Tr}\bar{\mathbf{D}} = \frac{1}{C}\int_0^C \mathrm{Tr}D_1 g(x_{co}(s), s)ds = \frac{2U_0}{E_0}, \tag{19}$$

where $U_0$ and $E_0$ are the energy gain in the cavity and beam energy respectively (See [6]). From (14), $\lambda^*_{2l} = \lambda_{2l-1}$ so that $\sum_1^3 \alpha_l = \mathrm{Tr}\bar{\mathbf{D}}$ and thus (19) is the Robinson sum rule, [10].

## 8. Remarks

**1.** The averages of $\mathbf{E}$ and $\mathbf{D}$ were computed under the non-resonance condition of (4). However the standard averaging error bound $|\mathbf{V}(s) - \mathbf{V}_a(s)| < O(\epsilon)$ for $0 \le s < O(1/\epsilon)$ requires a sufficient, $O(1)$, separation between $0, \nu_1, \nu_3, \nu_5$, and $1/2$. In the dissipative case, the $s$-interval of validity of the averaging approximation can be extended to all $s \ge 0$. Details are given in [**?**]. The resonant case is considered in [11].
**2.** The mean of $\vec{y}_L$ is easily handled and in the averaging approximation $\langle \vec{y}_L(s) \rangle \approx \mathbf{\Psi}(s) \exp(\epsilon \bar{\mathbf{D}} s) \mathbf{\Psi}^{-1}(0) \langle \vec{y}_L(0) \rangle$ and the covariance matrix is easily computed. Of course, in the dissipative case the mean approaches zero for large $s$. If $\vec{y}_L(0)$ is a Gaussian random vector, then $\vec{y}_L$ is a Gaussian process.
**3.** Spin-orbit motion in electron storage rings and especially spin diffusion due to quantum fluctuations can be treated approximately by introducing an 8-D matrix formalism. See Secs.2.7.7??? and 2.7.8???. The codes SLIM and SLICK discussed in Sec. 2.7.8??? are based on the orbital eigen-analysis of this section and can be used to calculate beam polarization, as well as the orbital dynamics of this section.
**4.** A general formalism for treating the linear electron beam dynamics with radiation effects taken into account is also presented in [12] and [13]. Whereas [12] starts from a kinetic description (Fokker-Planck equation) [13] uses (as we do) the SDE for the particle motion. The second order moments (beam envelopes) are calculated directly (rather than using the orbital-FSM) and these results are used in the computer code SAD. Furthermore generalized radiation integrals are derived which in the limiting case of a completely uncoupled machine reduce to the well known results of Sands, [14] (see Section 3.1.4.1???). This is also true in our case if we separate the six dimensional dynamics into fast betatron and slow synchrotron components via the dispersion (see Section 2.7.8???, p.181???). See [15] for a discussion of the beam envelope without radiation effects incorporated explicitly.

THINGS TO ADD:

(1) Put some of the remarks into the text.

## Appendix A. Derivation of EOM

## Appendix B. Resonance Case

## References

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