CSR by modal analysis

Introduction

To compute coherent synchrotron radiation (CSR) for particles on arbitrary planar orbits the usual approach is to work in space-time, invoking the free space Green function of the wave equation. This leads to nonlinear determinations of retarded times and integrations over singularities, and the resulting numerical problems can be controlled only with some difficulty. By contrast, a scheme based on Fourier analysis in spatial variables is entirely linear and without singularities, since the field equations reduce to the driven harmonic oscillator. The scheme is described for planar motion in bunch compressors with parallel conducting plates to represent the vacuum chamber. In this contribution we focus on the numerical solution of the wave equation for a fixed Gaussian source, i.e. for a distribution subject only to the field of the magnets, applying the Fourier method outlined above. Our ultimate goal is to develop a method to allow self-consistent evolution of the bunch of particles in a Vlasov framework where the numerical solution of the wave equation is a crucial step. In section 1 we describe the physical model and solve the wave equation for the electric field applying the Fourier method. In section 2 we discuss the numerical integration of the formula for the electric field derived in section 1. To reduce the computational time, the basic idea is to apply the method of stationary phase to the integration over the Fourier spatial variables \((k_z, k_x)\), due to the fact that the integrand has fast oscillations. From our numerical investigations we have seen that the method of stationary phase is only partially successful, i.e. can be applied only to some regions of the domain of integration, and not able to reduce significantly the computational time.

1 Fourier analysis of the wave equation

We solve the electromagnetic equations in the laboratory frame, where points are described by coordinates \((Z, X, Y)\). All particles move in planes parallel to the \((Z, X)\) plane (horizontal), and the beam is assumed to be bunched. For a bunch compressor, the \(Z\)-axis would be the beam line direction. We denote with \(\bar{E} = \gamma mc^2\) the energy of the reference particle. To represent the principal effect of the metallic vacuum chamber (suppression of CSR at long wave lengths), we place infinite parallel plates, perfectly conducting, at \(Y = \pm g\). The charge density of the beam has the form \(\rho(Z, X, Y, t) = H(Y)\Psi(R, t), \ R = (Z, X)\), where the vertical distribution \(H(Y)\) is arbitrary but fixed.

The 3D wave equation for the longitudinal and transverse components of the electric field
\[ \nabla^2 E - \frac{1}{c^2} E_{tt} = \frac{1}{\epsilon_0} \nabla \rho + \mu_0 J_t, \quad J = \rho \ v, \quad E = (E_Z, E_X). \]  

(1.1)

We make a Fourier development of the electric field as follows

\[ E(R, Y, u) = \int_{\mathbb{R}^2} dk \sum_{p=1}^{\infty} e^{ik \cdot R} \sin \alpha_p (Y + g) \hat{E}(k, \alpha_p, u). \]  

(1.2)

This meets the boundary conditions on the plates, \( E(Y = \pm g) = 0. \)

Taking the transform of (1.1) gives

\[ \hat{E}_{uu} + \kappa^2(k) \hat{E} = -\frac{i}{\epsilon_0} k \hat{\rho}(k, p, u) - \mu_0 c \hat{J}_u(k, \alpha_p, u) =: \hat{S}(k, p, u) \]  

(1.3)

\[ \kappa^2(k) = k^2 + \alpha^2_p, \quad u = ct. \]

Note that the wave equation in Fourier space is just a driven harmonic equation. The solution with zero initial conditions is

\[ E(R, Y, u) = \sum_{p=1}^{\infty} \sin \alpha_p (Y + g) \int_{\mathbb{R}^2} dk e^{ik \cdot R} \left[ \frac{1}{\kappa(k)} \sin \kappa(k) (u - v) \hat{S}(k, p, v) dv \right] \]  

(1.4)

Integration by parts on \( \hat{J}_u \) gives

\[ E(R, Y, u) = -Z_0 \sum_{p} \frac{1}{\sin \alpha_p (Y + g)} \int_{0}^{u} dv \int_{\mathbb{R}^2} dk e^{i k \cdot R} \left( \frac{\kappa(k)}{\kappa(k)} \hat{\rho}(k, p, v) \sin \kappa(k) (u - v) \right) \]  

(1.5)

where

\[ \hat{\rho}(k, p) = \frac{1}{(2\pi)^2 g} \int_{-g}^{g} dR \int_{-g}^{g} dY e^{-i k \cdot R} \sin \alpha_p (Y + g) \rho(R, Y) dR dY, \]  

(1.6)

\( \hat{J} \) just replaces \( \rho \) by \( J \) and \( H_p = \frac{(-1)^{p+1}}{2} e^{-\frac{1}{2} (\alpha_p \sigma Y)^2}. \)

To calculate \( \rho \) and \( J \) we need the orbits which are given by

\[ Z(u) = \varphi_1(u, \gamma(Z_0)), \quad \varphi_1(0, \gamma(Z_0)) = Z_0 \]
\[ X(u) = x_0 + \varphi_2(u, \gamma(Z_0)), \quad \varphi_2(0, \gamma(Z_0)) = 0 \]  
\[ Y(u) = Y_0 \]  

We take the initial density of the form
\[ \rho_0(Z_0, X_0, Y_0) = f(X_0, Y_0)g(Z_0). \]  

Defining the transformation
\[ (\mathbf{R}, Y) = (0, X_0) + \varphi(u, \gamma(Z)) \]  
in (1.6) gives
\[ \hat{\rho}(k, p, u) = \frac{1}{(2\pi)^2 g} \int e^{-ik_{\mathbf{X}}_0} e^{-ik \cdot \varphi(u, \gamma(Z_0))} \sin \alpha p (Y_0 + g) \right. \]
\[ \left. \times f(X_0, Y_0)g(Z_0) dZ_0 dX_0 dY_0 \right. \]  
where we have used the fact that
\[ \rho(\mathbf{R}, Y, u) \left| \frac{\partial(Z, X, Y)}{\partial(Z_0, X_0, Y_0)} \right| = \rho_0(\mathbf{R}_0, Y_0) = Q f(X_0, Y_0)g(Z_0) \]

It follows that
\[ \hat{\rho}(k, p, u) = Q \hat{f}(k_{\mathbf{X}}, p) \frac{1}{2\pi} \int e^{-ik \cdot \varphi(u, \gamma(Z_0))} g(Z_0) dZ_0 \]
\[ = Q \hat{f}(k_{\mathbf{X}}, p) \frac{1}{2\pi} \int e^{-ik \cdot \varphi(u, \gamma)} \mu(\gamma) d\gamma \]

where \( \mu \) is the (initial) energy density.

The current density is
\[ J(\mathbf{R}, Y, u) = \rho(\mathbf{R}, Y, u) \mathbf{V}(\mathbf{R}, u) = \rho(\mathbf{R}, Y, u) c \hat{\varphi}(u, \gamma(Z_0(Z))) \]

From (1.6) with \( \rho \) replaced by \( J \) and using the transformation (1.9) we obtain
\[ \hat{J}(k, p, u) = \frac{Qc}{2\pi} \hat{f}(k_{\mathbf{X}}, p) \int e^{-ik \cdot \varphi(u, \gamma(z_0))} \hat{\varphi}(u, \gamma(Z_0)) g(Z_0) dZ_0 \]
\[ = \frac{Qc}{2\pi} \hat{f}(k_{\mathbf{X}}, p) \int e^{-ik \cdot \varphi(u, \gamma)} \hat{\varphi}(u, \gamma) \mu(\gamma) d\gamma \]
To make (1.12) and (1.14) more tractable we linearize the orbit with respect to \( \gamma \) around \( \bar{\gamma} \)

\[
\begin{align*}
\varphi(u, \gamma) &\approx \varphi(u, \bar{\gamma}) + \varphi_\gamma(u, \bar{\gamma})(\gamma - \bar{\gamma}) \\
\dot{\varphi}(u, \gamma) &\approx \dot{\varphi}(u, \bar{\gamma}) + \dot{\varphi}_\gamma(u, \bar{\gamma})(\gamma - \bar{\gamma})
\end{align*}
\] (1.15)

Defining \( \mu(\xi) = \mu(\xi + \bar{\gamma}) \) we obtain

\[
\hat{\rho}(k, p, u) = Q e^{-ik \cdot \varphi(u, \bar{\gamma})} \hat{f}(k_X, p) \hat{\mu}_c(k \cdot \varphi_\gamma(u, \bar{\gamma}))
\] (1.16)

and

\[
\hat{J}(k, p, u) = Q e^{-ik \cdot \varphi(u, \bar{\gamma})} \hat{f}(k_X, p) \int_{-\infty}^{\infty} e^{-i(k \cdot \varphi_\gamma(u, \bar{\gamma})) \xi} (\dot{\varphi}(u, \bar{\gamma}) + \dot{\varphi}_\gamma(u, \bar{\gamma}) \xi) \mu_c(\xi) d\xi
\]

\[
= Q e^{-i(k \cdot \varphi(u, \bar{\gamma}))} \hat{f}(k_X, p) \left[ \dot{\varphi}(u, \bar{\gamma}) \hat{\mu}_c(k \cdot \varphi_\gamma(u, \bar{\gamma})) - i \hat{\dot{\varphi}}_\gamma(u, \bar{\gamma}) \hat{\mu}_c'(k \cdot \varphi_\gamma(u, \bar{\gamma})) \right]
\]

(1.17)

Plugging (1.16) and (1.17) in (1.5) we obtain

\[
E(R, Y, u) = -Z_0 Q c \sum_p \sin \alpha_p(Y + g) \int_0^u dv \int_{\mathbb{R}^2} dke^{ik \cdot (R - \varphi_\gamma(v, \bar{\gamma}))} \hat{f}(k_X, p)
\times \left[ \frac{k}{\kappa(k)} \mu_c(k \cdot \varphi_\gamma(v, \bar{\gamma})) \sin \kappa(k)(u - v) + \left\{ \dot{\varphi}(v, \bar{\gamma}) \hat{\mu}_c(k \cdot \varphi_\gamma(v, \bar{\gamma})) \right\} \cos \kappa(k)(u - v) \right]
\]

(1.18)

Now we assume the Gaussian source model

\[
f(X) = \frac{1}{\sqrt{2\pi} \sigma_X} e^{-\frac{x^2}{2\sigma_X^2}}, \quad \mu_c(\xi) = \frac{1}{\sqrt{2\pi} \sigma_\gamma} e^{-\frac{\xi^2}{2\sigma_\gamma^2}}
\]

(1.19)

so

\[
\hat{f}(k_X, p) = e^{-\frac{1}{2}\sigma_X^2 k_X^2} H_p, \quad \hat{\mu}_c(k_\xi) = e^{-\frac{1}{2}\sigma_\gamma^2 k_\xi^2}
\]

(1.20)

which gives

\[
E(R, Y, u) = -\frac{Z_0 Q c}{2\pi} \sum_p H_p \sin \alpha_p(Y + g) \int_0^u dv \int_{\mathbb{R}^2} dke^{-\frac{1}{2}(\sigma_X^2 k_X^2 + \sigma_\gamma^2 (k \cdot \varphi_\gamma(v, \bar{\gamma})))^2}
\]


\[
\times e^{i k \cdot (R - \varphi(v, \bar{\gamma}))} \left[ i \frac{k}{\kappa} \sin \kappa (u - v) \right] \\
+ \left\{ \varphi(v, \bar{\gamma}) + i \dot{\varphi}_\gamma(v, \bar{\gamma}) \sigma_\gamma^2 (k \cdot \varphi(v, \bar{\gamma})) \right\} \cos \kappa (u - v) \right] (1.21)
\]

2 Numerical test: method of stationary phase

We are interested in the longitudinal component of the electric field \( \mathcal{E} \) at time \( u \) evaluated at the position of the reference particle at time \( u + \Delta u \)

\[
\mathcal{E} = \varphi(u + \Delta u, \bar{\gamma}) \cdot E(\varphi(u + \Delta u, \bar{\gamma}), Y, u) \\
= -\frac{Z_0 Q_c}{2\pi} \sum_{p} H_p \sin \alpha_p (Y + g) \int_{0}^{u} dv \int_{\mathbb{R}^2} dk e^{-\frac{1}{2} (\sigma^2_{\bar{\gamma} \cdot \bar{\gamma}} + \sigma^2_{\gamma} (k \cdot \varphi(v, \bar{\gamma})^2)} \\
\times e^{i k \cdot (\varphi(u + \Delta u, \bar{\gamma}) - \varphi(v, \bar{\gamma}))} \varphi(u + \Delta u, \bar{\gamma}) \cdot \left[ i \frac{k}{\kappa} \sin \kappa (u - v) \\
+ \left\{ \varphi(v, \bar{\gamma}) + i \dot{\varphi}_\gamma(v, \bar{\gamma}) \sigma_\gamma^2 (k \cdot \varphi(v, \bar{\gamma})) \right\} \cos \kappa (u - v) \right] (2.22)
\]

For reality \( (\Im(\mathcal{E}) = 0) \) it follows that

\[
\mathcal{E} = -\frac{Z_0 Q_c}{2\pi} \sum_{p} H_p \sin \alpha_p (Y + g) \int_{0}^{u} dv \int_{\mathbb{R}^2} dk e^{-\frac{1}{2} (\sigma^2_{\bar{\gamma} \cdot \bar{\gamma}} + \sigma^2_{\gamma} (k \cdot \varphi(v, \bar{\gamma})^2)} \\
\times \varphi(u + \Delta u, \bar{\gamma}) \cdot \left[ \dot{\varphi}(v, \bar{\gamma}) \cos k \cdot a \cos \kappa b - \frac{k}{\kappa} \sin (k \cdot a) \cos \kappa b \\
+ \dot{\varphi}_\gamma(v, \bar{\gamma}) \sigma_\gamma^2 (k \cdot \varphi(v, \bar{\gamma})) \sin (k \cdot a) \cos \kappa b \right] (2.23)
\]

where \( a = \varphi(u + \Delta u, \bar{\gamma}) - \varphi(v, \bar{\gamma}) \) and \( b = u - v \). In polar coordinates

\[
k_z = k \cos \theta \\
k_x = k \sin \theta
\]

the p-mode of \( \mathcal{E} \) reads (leaving out \( H_p \sin \alpha_p (Y + g) \))

\[
\mathcal{E}_p = -\frac{Z_0 Q_c}{2\pi} \int_{0}^{u} dv \int_{0}^{\infty} dk \int_{-\pi}^{\pi} d\theta e^{-\frac{k^2}{2} \left[ \sigma^2_{\bar{\gamma} \cdot \bar{\gamma}} + \sigma^2_{\gamma} (Z_{\bar{\gamma}}(v, \bar{\gamma}) \cos \theta + X_{\bar{\gamma}}(v, \bar{\gamma}) \sin \theta)^2 \right]} \\
\times \left[ \dot{\varphi}(u + \Delta u, \bar{\gamma}) \cdot \dot{\varphi}(v, \bar{\gamma}) \cos k (a_z \cos \theta + a_x \sin \theta) \cos \kappa b \right]
\]
\[-\frac{k}{\kappa} \left( \tilde{Z}(u + \Delta u, \gamma) \cos \theta + \tilde{X}(u + \Delta u, \gamma) \sin \theta \right) \sin k(a_z \cos \theta + a_x \sin \theta) \sin \kappa b \]
\[
+ \dot{\phi}(u + \Delta u, \gamma) \cdot \dot{\varphi}_i(v, \gamma) k \left( \cos \theta Z_i(v, \gamma) + \sin \theta X_i(v, \gamma) \right) \times \sigma^2 \sin k(a_z \cos \theta + a_x \sin \theta) \cos \kappa b \]
\]

\[\mathcal{E}_p \text{ can be written as } \mathcal{E}_{p-} + \mathcal{E}_{p+} \text{ where} \]
\[
\mathcal{E}_{p-} = -\frac{Z_0 Qc}{2\pi} \int_0^u dv \int_0^\infty dk d\theta e^{-\frac{k^2}{4\kappa^2} \left[ \sigma^2 \cos^2 \theta + \sigma_1^2 (Z(v, \gamma) \cos \theta + X(v, \gamma) \sin \theta)^2 \right]} \]
\[
\times \frac{1}{2} \left[ \dot{\phi}(u + \Delta u, \gamma) \cdot \dot{\varphi}(v, \gamma) \cos(\phi - \kappa b) \right] - \frac{k}{\kappa} \left( \tilde{Z}(u + \Delta u, \gamma) \cos \theta + \tilde{X}(u + \Delta u, \gamma) \sin \theta \right) \cos(\phi - \kappa b) \]
\[
+ \dot{\phi}(u + \Delta u, \gamma) \cdot \dot{\varphi}_i(v, \gamma) k \left( \cos \theta Z_i(v, \gamma) + \sin \theta X_i(v, \gamma) \right) \sigma^2 \sin(\phi - \kappa b) \]  
(2.24)

\[
\mathcal{E}_{p+} = -\frac{Z_0 Qc}{2\pi} \int_0^u dv \int_0^\infty dk d\theta e^{-\frac{k^2}{4\kappa^2} \left[ \sigma^2 \cos^2 \theta + \sigma_1^2 (Z(v, \gamma) \cos \theta + X(v, \gamma) \sin \theta)^2 \right]} \]
\[
\times \frac{1}{2} \left[ \dot{\phi}(u + \Delta u, \gamma) \cdot \dot{\varphi}(v, \gamma) \cos(\phi + \kappa b) \right] + \frac{k}{\kappa} \left( \tilde{Z}(u + \Delta u, \gamma) \cos \theta + \tilde{X}(u + \Delta u, \gamma) \sin \theta \right) \cos(\phi + \kappa b) \]
\[
+ \dot{\phi}(u + \Delta u, \gamma) \cdot \dot{\varphi}_i(v, \gamma) k \left( \cos \theta Z_i(v, \gamma) + \sin \theta X_i(v, \gamma) \right) \sigma^2 \sin(\phi + \kappa b) \]  
(2.25)

and \( \phi = k(a_z \cos \theta + a_x \sin \theta) \).

The strategy we follow to evaluate \( \mathcal{E}_p \) is to apply the method of stationary phase in \( \theta \) to \( \mathcal{E}_{p-}, \mathcal{E}_{p+} \) and in \( k \) to \( \mathcal{E}_{p+} \) for different regions of the support \( (k, \theta) \) (see chapter 4 of the book by E. T. Copson Asymptotic Expansions, Cambridge University Press, 1965). Applying the method of stationary phase in \( \theta \) (Copson, eq. (14.3)) we get

\[
\mathcal{E}_{p-} = -\frac{Z_0 Qc}{\sqrt{2\pi a} \kappa} \int_{0}^{u} dv \int_{0}^{\infty} dk \sqrt{k} e^{-\frac{k^2}{4\kappa^2} \left[ \sigma^2 \cos^2 \theta + \sigma_1^2 (a_z Z(v, \gamma) + a_x X(v, \gamma))^2 \right]} \]
\[
\times \frac{1}{2} \left[ \dot{\phi}(u + \Delta u, \gamma) \cdot \dot{\varphi}(v, \gamma) \cos(\phi - \kappa b - \frac{\pi}{4}) \right] - \frac{k}{\kappa} \left( \tilde{Z}(u + \Delta u, \gamma) \dot{a}_z + \tilde{X}(u + \Delta u, \gamma) \dot{a}_x \right) \cos(\phi - \kappa b - \frac{\pi}{4}) \]
\[
+ \dot{\phi}(u + \Delta u, \gamma) \cdot \dot{\varphi}_i(v, \gamma) \left( \dot{a}_z Z_i(v, \gamma) + \dot{a}_x X_i(v, \gamma) \right) \sigma^2 \sin(\phi - \kappa b - \frac{\pi}{4}) \]  
(2.27)

and
\[ \mathcal{E}_{p,v} = -\frac{Z_0 Q e}{2\pi a} \int_0^u \int_0^\infty \frac{dv}{d\theta k e} e^{-\frac{v^2}{2} [\sigma_\phi^2 \phi^2 + \sigma_\gamma^2 (\gamma \cos \theta + \phi \sin \theta)^2]} \]

\[ \times \frac{1}{2} \left[ \dot{\phi}(u + \Delta u, \bar{\gamma}) \cdot \dot{\phi}(\gamma, \bar{\gamma}) \right. \]

\[ + \frac{k}{\kappa} \left. \left( \dot{Z}(u + \Delta u, \bar{\gamma}) \dot{\phi}(\gamma, \bar{\gamma}) \right) \right]_{k=k_{\text{min}}} \]

\[ \phi = a \frac{u}{a}. \]

Applying the method of stationary phase in \( k \) (Copson, eq.e (14.2)) we get

\[ \mathcal{E}_{p,v} = \frac{Z_0 Q e}{2\pi} \int_0^u \int_{-\pi}^\pi \frac{dv}{d\theta k e} e^{-\frac{v^2}{2} [\sigma_\phi^2 \phi^2 + \sigma_\gamma^2 (\gamma \cos \theta + \phi \sin \theta)^2]} \]

\[ \times \frac{1}{2} \left[ \dot{\phi}(u + \Delta u, \bar{\gamma}) \cdot \dot{\phi}(\gamma, \bar{\gamma}) \right. \]

\[ + \frac{k}{\kappa} \left. \left( \dot{Z}(u + \Delta u, \bar{\gamma}) \cos \theta + \dot{X}(u + \Delta u, \bar{\gamma}) \sin \theta \right) \right]_{k=k_{\text{min}}} \]

where \( \psi = a \frac{u}{a} + \frac{ib}{\kappa}. \)

We calculated the v-integrand of \( \mathcal{E}_p \) for \( p=1,3,5,7,9 \) at the position of the reference particle, i.e. \( \Delta u = 0 \), at \( u=22.911 \) (see figure 1).

The strategy we applied to calculate \( \mathcal{E}_p \) consists to partition the \( (k, \theta) \) region as follows (see figure 2 for the support of \( \mathcal{E}_p \) defined by the Gaussian cutoff

\[ \exp(-k^2/2f(\theta)) = \exp(-10), \text{ i.e. } k = \sqrt{20/f(\theta)}; \]

- for \( k \leq 1000 \) and \( v \leq 22.87 \) we integrated \( \mathcal{E}_p \) in the range \( 0 \leq k \leq k_{\text{sp}0}, -\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2} \) with a two-dimensional integrator (adaptive code based on the Gauss-Kronrod formula) while in the range \( k_{\text{sp}0} \leq k \leq 1000 \) we applied the method of stationary phase in theta. The value of \( k_{\text{sp}0} \) we used is 100/60.

- for \( k \geq 1000 \) and \( v \leq 22.87 \) we calculated \( \mathcal{E}_{p+} \) and \( \mathcal{E}_{p-} \) as follows:
  - We've integrated \( \mathcal{E}_{p+} \) applying the method of stationary phase in \( k \).
  - For the integration of \( \mathcal{E}_{p-} \), in the range \( 1000 \leq k \leq k_{sp0} \) we did the two-dimensional integration (Gauss-Kronrod code) while in the range \( k_{sp0} \leq k \leq 2000 \) we applied the method of stationary phase in theta.
  - For \( k \geq 2000 \), \(-\frac{\pi}{3} \leq \theta \leq \frac{\pi}{3} \) we integrated \( \mathcal{E}_{p-} \) with the two-dimensional integrator (Gauss-Kronrod code).
For $k \geq 2000$, $-\frac{\pi}{2} \leq \theta \leq -\frac{\pi}{3} \cup \frac{\pi}{3} \leq \theta \leq \frac{\pi}{2}$ and $v \leq 22.87$ we’ve seen the integration has a negligible contribution.

- Finally for $22.87 \leq v \leq 22.91$ we did the two-dimensional integration (Gauss-Kronrod code).

The important parameters are $k = 1000$, the minimum value of $k$ for which the method of stationary phase in $k$ can be applied to $E_{p,\pm}$ and $k = 2000$, the maximum value of $k$ for which the method of stationary phase in $\theta$ can be applied to $E_p$. In the figures after the figure 6 we have plotted $E_p$, $E_{p,\pm}$ for $p=1,5,9$ and $v=22.41$, 22.61, 22.78, 22.88 for different values of $k$ and $\theta$. It seems that for $k \geq 1000$ and $v \leq 22.88$ the oscillations are fast enough to apply the method of stationary phase in $k$. For $k \geq 2000$ the method of stationary phase in $\theta$ is not accurate because the asymmetry given by the Gaussian.

In figure 8 is plotted $E_1(v=22.61, k, \theta = 0)$. The integrand has fast oscillations but the profile is irregular because the interference between $E_{1+}$ and $E_{1-}$ (figures 9 and 10). We have seen that $E_{1\pm}$ can be evaluated applying the method of stationary phase without stationary point ($E_{1\pm}$ has a stationary point $k_{stat} = \alpha_p |\beta|/\sqrt{b^2 - \beta^2}$ when $\beta = a_z \cos \theta + a_x \sin \theta$ is negative. Because $a_z \gg a_x$, $\beta$ is negative only for $\theta$ near to $\frac{\pi}{2}$). As it was previously pointed out, the value of $k_{min}$ (minimum value of $k$ to apply the method of stationary phase in $k$) has been chosen equal to 1000 for every $\theta$ and $v$. A better choice would be $k_{min} = k_{min}(v, \theta)$. From figure 8 one can understand that $k_{min}$ can be smaller than 1000. In figure 11, 12, 13...
Fig. 2. Support of $E_p$ for different values of $v$ defined by the Gaussian cutoff $\exp(-k^2/2f(\theta)) = \exp(-10)$, i.e. $k = \sqrt{20/f(\theta)}$.

Fig. 3. k-stationary point for $E_{p-}$: $k_{stat} = \alpha_p\beta/\sqrt{b^2 - \beta^2}$, where $\beta = a_z\cos \theta + a_x\sin \theta$. Left: $v=22.41$. Right: $v=22.61$

Fig. 4. k-stationary point for $E_{p-}$. Left: $v=22.78$. Right: $v=22.88$
are shown $\mathcal{E}_1$, $\mathcal{E}_{1-}$ and $\mathcal{E}_{1+}$ respectively for $v=22.61$ and $\theta = 0.0628$. We see that the range of integration of $k$ is reduced by a factor $3$. $\mathcal{E}_{1+}$ has still fast oscillations. In figure 14, 15, 16 are shown $\mathcal{E}_1$, $\mathcal{E}_{1-}$ and $\mathcal{E}_{1+}$ respectively for $v=22.61$ and $\theta = 0.628$. The interference between $\mathcal{E}_{1-}$ and $\mathcal{E}_{1+}$ is clearly visible in figure 14. The oscillations of $\mathcal{E}_{1+}$ are smaller with respect to figure 13 ($\theta = 0.0628$). $\mathcal{E}_{1-}$ starts to show fast oscillations. This suggests the possibility to evaluate it applying the method of stationary phase without stationary point. Let us note that the value of the stationary point is near 400 (see figure 12, right frame, for an enlargement). In figure 17, 18, 19 are shown $\mathcal{E}_1$, $\mathcal{E}_{1-}$ and $\mathcal{E}_{1+}$ respectively for $v=22.61$ and $\theta = \frac{\pi}{2}$. The profile of $\mathcal{E}_1$ has now the maximum irregularity. $\mathcal{E}_{1-}$ and $\mathcal{E}_{1+}$ show the same oscillations, that seem fast enough to apply the method of stationary phase. In our strategy to evaluate $\mathcal{E}_1$ we have neglected the contribution from $\mathcal{E}_{1-}$ in the region $-\frac{\pi}{2} \leq \theta \leq -\frac{\pi}{3} \cup \frac{\pi}{3} \leq \theta \leq \frac{\pi}{2}$. This can be proved applying the method of stationary phase. We didn’t show the same figures for $-\frac{\pi}{2} \leq \theta \leq 0$ because by symmetry $\mathcal{E}_1$, $\mathcal{E}_{1-}$ and $\mathcal{E}_{1+}$ are almost invariant under the transformation $\theta \rightarrow -\theta$.

In figure 20, 21, 22 we now show $\mathcal{E}_1$ for $v=22.61$ and different values of $k$ ($k=10,50,150$). In the right frame of every figure the integrand has been put in the form $kg(k, \theta) \exp(-\frac{k}{2}f(\theta))$. In red is plotted the term $k \exp(-\frac{k}{2}f(\theta))$, in blue the term $kg(k, \theta)$. The value of $k_{stat}$ (minimum value of $k$ to apply the method of stationary phase in $\theta$) is $k_{stat} = \frac{100}{60} = 55.54$. The stationary point is $\theta_{stat} = -\arctan \frac{4\pi}{3} = -0.0027$. In figure 23, 24, 25 is shown $\mathcal{E}_1$ for $v=22.61$ and $k=300,600,2000$. We have seen that $k=2000$ is the maximum value for which the integration in $\theta$ can be done applying the method of stationary phase. In figure 26, 27, 28 is shown $\mathcal{E}_1$ for $v=22.61$ and $k=5000,10000,20000$. In the left frame of the figures one can see that the asymmetry given by the Gaussian (not centered at the stationary point) becomes important. As a consequence the method of stationary phase becomes inaccurate (the formula evaluates the integrand at the stationary point, so cannot “see” the asymmetry). A close look at the figures suggests the possibility to apply the method of stationary phase without stationary points “outside” the stationary point where the oscillations are fast enough, leaving the two-dimensional integrator to the region “around” the stationary point. The value $k=2000$ chosen is independent on $v$ (and $p$). A better understanding of the relation between the “size” of the region around the stationary point where the integrand is slowly varying and the variance of the Gaussian should give the dependence of $k$ on $v$ (and $p$). The asymmetry given by the Gaussian is clearly visible in figure 29. Let us note that the region of integration has been strongly reduced by the Gaussian cutoff.

In figure 31, 32, 33 is shown $\mathcal{E}_{p-}$ for $v=22.41$, $\theta = 0$ and $p=1,5,9$ respectively. One can clearly see that the region in $k$ where the integrand has fast oscillations increases as $p$ increases. Remembering that $\mathcal{E}_{p-}$ has been integrated for $k \leq 2000$ applying the method of stationary phase in $\theta$, the evaluation of $\mathcal{E}_{p-}$ for $k \geq 2000$ could be done, until some value of $k$, applying the method of stationary phase in $k$, at least for the higher values of $p$. In figure 34, 35, 36 is shown $\mathcal{E}_{p-}$ for $v=22.41$, $\theta = \frac{\pi}{2}$ and $p=1,5,9$ respectively. Let us note that the stationary point in $k$ ($k_{stat} = a_p/b/\sqrt{\beta^2 - \beta^2}$, where $\beta = a_x \cos \theta + a_y \sin \theta$, $a_p = \frac{\pi}{3}$), an increasing function of $p$, is smaller then 2000. One still has the possibility to integrate in $k$ from 2000 to some $k$ applying the method of stationary phase. In figure 37, 38, 39 is shown $\mathcal{E}_{p-}$ for $v=22.41$, $\theta = \frac{\pi}{2}$ and $p=1,5,9$ respectively. Let us remember that in the region $-\frac{\pi}{3} \leq \theta \leq -\frac{\pi}{3} \cup \frac{\pi}{3} \leq \theta \leq \frac{\pi}{2}$ we found that $\mathcal{E}_{p-}$ gives a negligible contribution (to be verified applying the method of stationary phase in $k$ without stationary point). We note that the oscillations decrease as $p$.
increases and that the oscillations for \( p = 1 \) are faster than the oscillations for \( p = 1, v = 22.61 \) (figure 18). In figure 40, 41, 42 is shown \( \mathcal{E}_{p+} \) for \( v = 22.41, \theta = 0 \) and \( p = 1, 5, 9 \) respectively. The oscillations still decrease as \( p \) increases and for \( p = 1 \) are faster then the oscillations for \( p = 1, v = 22.61 \) (figure 19). Let us note the different range in \( k \) of the right frames. In figure 43, 44, 45 is shown \( \mathcal{E}_{p+} \) for \( v = 22.41, \theta = \frac{\pi}{3} \) and \( p = 1, 5, 9 \) respectively while in figure 46, 47, 48 is shown \( \mathcal{E}_{p+} \) for \( v = 22.41, \theta = \frac{\pi}{2} \). We note again the same dependence on \( p \) and for \( \theta = \frac{\pi}{2} \) the oscillations for \( p = 1 \) are faster then the oscillations for \( p = 1, v = 22.61 \).

In figure 49, 50, 51 is shown \( \mathcal{E}_{p-} \) for \( v = 22.78, \theta = 0 \) and \( p = 1, 5, 9 \) respectively. These figures should be compared with the figure 31, 32, 33 \((\mathcal{E}_{p-} \text{ for } v = 22.41, \theta = 0 \text{ and } p = 1, 5, 9)\). In figure 52, 53, 54 is shown \( \mathcal{E}_{p-} \) for \( v = 22.78, \theta = \frac{\pi}{3} \) and \( p = 1, 5, 9 \) respectively. These figures should be compared with the figure 34, 35, 36 \((\mathcal{E}_{p-} \text{ for } v = 22.41, \theta = \frac{\pi}{3} \text{ and } p = 1, 5, 9)\) and 37, \( \mathcal{E}_{p-} \text{ for } v = 22.41, \theta = \frac{\pi}{2} \) and \( p = 1, 5, 9 \). Let us note again the stationary points in \( k \) for \( \theta = \frac{\pi}{3} \) and that the oscillations for \( v = 22.78 \) are slower then the oscillations for \( v = 22.41 \).

In figure 52, 53, 54 is shown \( \mathcal{E}_{p+} \) for \( v = 22.78, \theta = 0 \) and \( p = 1, 5, 9 \) respectively, to be compared with the figure 40, 41, 42 \((\mathcal{E}_{p+} \text{ for } v = 22.41, \theta = 0 \text{ and } p = 1, 5, 9)\). Let us note the different range in \( k \) for \( p = 9, v = 22.78 \). In figure 58, 59, 60 is shown \( \mathcal{E}_{p+} \) for \( v = 22.78, \theta = \frac{\pi}{3} \) and \( p = 1, 5, 9 \) respectively while in figure 61, 62, 63 is shown \( \mathcal{E}_{p+} \) for \( v = 22.78, \theta = \frac{\pi}{2} \) and \( p = 1, 5, 9 \) respectively. These figures should be compared with the figure 43, 44, 45 \((\mathcal{E}_{p+} \text{ for } v = 22.41, \theta = \frac{\pi}{3} \text{ and } p = 1, 5, 9)\) and 46, 47, 48 \((\mathcal{E}_{p+} \text{ for } v = 22.41, \theta = \frac{\pi}{2} \text{ and } p = 1, 5, 9)\). Let us note how slower are the oscillations for \( v = 22.78, \theta = \frac{\pi}{2} \) with respect to \( v = 22.41, \theta = \frac{\pi}{2} \).

In figure 64, 65, 66 is shown \( \mathcal{E}_{p-} \) for \( v = 22.88, \theta = 0 \) and \( p = 1, 5, 9 \) respectively while in figure 67, 68, 69 \( \mathcal{E}_{p-} \) for \( v = 22.88, \theta = \frac{\pi}{3}, \frac{\pi}{2} \) and \( p = 1, 5, 9 \). Let us note the the oscillations of \( \mathcal{E}_{p-} \) for \( v = 22.88 \) are slower than the oscillations for the previous values of \( v \). In figure 70, 71, 72 is shown \( \mathcal{E}_{p+} \) for \( v = 22.88, \theta = 0 \) and \( p = 1, 5, 9 \) respectively while in figure 73, 74, 75 \( \mathcal{E}_{p+} \) for \( v = 22.88, \theta = \frac{\pi}{3}, \frac{\pi}{2} \) and \( p = 1, 5, 9 \). The value \( v = 22.88 \) is in the range of integration of \( v \) for which we did the two-dimensional integration \((v \geq 22.87)\). This point can be understood looking at figures 73, 74, 75 where the oscillations are not fast enough. One could nevertheless apply the method of stationary phase in \( k \) without stationary point to evaluate \( \mathcal{E}_{p+} \) not to the full integration region \(-\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2}\) but to a smaller one around \( \theta = 0 \), where the oscillations are fast enough.

In figure 76 is shown \( \mathcal{E}_{p-} \) for \( v = 22.41, k = 2000, p = 1, 5 \). In figure 77 is shown \( \mathcal{E}_{p-} \) for \( v = 22.41, k = 2000, p = 9 \) (left frame) and \( \mathcal{E}_{p-} \) for \( v = 22.78, k = 2000, p = 1 \) (right frame). In figure 78 is shown \( \mathcal{E}_{p+} \) for \( v = 22.78, k = 2000, p = 1, 5, 9 \). One can understand that for \( k \geq 2000 \) and not too large the two-dimensional integration of \( \mathcal{E}_{p+} \) is very time consuming, in particular for \( v = 22.41 \). As it was previously mentioned, a good strategy would be to apply the method of stationary phase in \( \theta \) with no stationary points “outside” the stationary point integrating in two-dimension only “around” it. In figure 79, 80 (left frame) is shown \( \mathcal{E}_{p-} \) for \( v = 22.88, k = 2000 \) and \( p = 1, 5, 9 \). We see that the oscillations are not so fast for this value of \( v \). In the figure 80 (right frame), 81 is shown \( \mathcal{E}_{p} \) for \( v = 22.78, k = 130 \) and \( p = 1, 5, 9 \). \( k = 130 \) is the value of \( k_{sp\theta} = \frac{100}{62} \). In figure 82, 83 (left frame) is shown \( \mathcal{E}_{p} \) for \( v = 22.88, k = 462 \) and \( p = 1, 5, 9 \). In figure 83 (right frame), 84 is shown \( \mathcal{E}_{p} \) for \( v = 22.41, k = 33 \). We want now to point out what could be done to speed up the evaluation of \( \mathcal{E}_{p} \). To improve the strategy one should find \( k_{min} \) (method of stationary phase in \( k \) without stationary point) as a function of \((v, \theta)\) and \( k_{sp\theta} \) (method of stationary phase in \( \theta \) as a function of \( v \)). We noted
that another possibility to speed up the calculation of $E_p$ would be to apply the method of stationary phase in $\theta$ (first formula, no stationary point) for $k \geq 2000$ to integrate in $\theta$ “outside” the stationary point, integrating in two-dimension only “around” the stationary point.

It seems that the two-dimensional integration cannot be avoided for $v \geq 22.87$ because the oscillations are not fast enough (see the figures for $v=22.88$).

Finally, we give an explicit example to understand the reason why the method of stationary phase in $\theta$ cannot be applied for large values of $k$. Let us consider the case $v=22.61$ and $k = 5032.13$ ($k_{\min} = 55.54, k_{\max} = 141258.3956$).

The values of the parameters that are independent on $k$ and $\theta$ read

\[
\begin{align*}
\sigma_{x_0}^2 &= 10^{-6} \\
\sigma_\gamma^2 &= 4618.56 \\
Z_\gamma(v, \gamma) &= -3.35313315 \times 10^{-7} \\
X_\gamma(v, \gamma) &= -1.09951354 \times 10^{-7} \\
\varphi(u + \Delta u, \gamma) \cdot \dot{\varphi}(v, \gamma) &= 0.999864552 \\
a_z &= 0.299972909 \\
a_x &= -0.00823070382 \\
\dot{Z}((u + \Delta u, \gamma) &= 0.999864558 \\
\dot{X}((u + \Delta u, \gamma) &= -0.016457617 \\
\dot{\varphi}(u + \Delta u, \gamma) \cdot \dot{\varphi}(v, \gamma) &= 6.08920383 \times 10^{-8}
\end{align*}
\]

The integrand of (2.24) is plotted in figure 5, where the factor $-cQZ_0/2\pi$ has been omitted. Keeping only the dominant terms we now approximate the integrand of (2.24) with the function (see figure 5,6)

\[
f(\theta) = 48112 \cos(15104 \cos(\theta + 0.0274)) e^{-[1267 \sin^2(\theta) + 0.658 \cos^2(\theta)]}
\]

The argument of $\cos(15104 \cos(\theta + 0.0274))$ has a stationary point at $x = -0.0274$

The numerical integration of $f(\theta)$ from $\pi - 0.0274$ to 3.2 reads

\[
\int_{\pi - 0.0274}^{3.2} f(\theta) d\theta = -35.564 \quad (2.30)
\]

while the method of stationary phase gives

\[
\int_{\pi - 0.0274}^{3.2} f(\theta) d\theta \approx 48112 \sqrt{\frac{\pi}{30208}} e^{-[1267 \sin^2(\pi - 0.0274) + 0.658 \cos^2(\pi - 0.0274)]}
\]
in complete disagreement with (2.30).
An alternative way to evaluate (2.24) is to apply a code for fast oscillating integrals. We note that

\[
\int_0^{2\pi} e^{ika \cos \theta} f(k, \theta) d\theta = 2 Re \int_0^{\pi} e^{ika \cos \theta} f(k, \theta) d\theta
\]

Putting \( \cos \theta = x \) and killing the singularity of the Jacobian we get

\[
\int_{-1}^{1} e^{ikax} g(k, x) \frac{1}{\sqrt{1-x^2}} dx = \int_{-1}^{1} \frac{e^{ikax}}{\sqrt{1-x^2}} \left[ g(k, x) - \frac{(1+x)}{2} g(k, 1) - \frac{(1-x)}{2} g(k, -1) \right] dx
\]

\[
+ \frac{g(k, 1)}{2} \int_{-1}^{1} \frac{e^{ikax} (1+x)}{\sqrt{1-x^2}} dx + \frac{g(k, -1)}{2} \int_{-1}^{1} \frac{e^{ikax} (1-x)}{\sqrt{1-x^2}} dx
\]  

(2.32)
where \( g(k, x) = f(k, \cos^{-1} x) \). If we were able to prove that the first term in the right hand side of (2.32) gives a negligible contribution, we would evaluate only the second term in the right hand side related to the Bessel functions

\[
\int_{-1}^{1} \frac{e^{ikax}}{\sqrt{1-x^2}} dx = \int_{0}^{\pi} \cos(ka \cos \theta) d\theta = \pi J_0(ka)
\]

\[
\int_{-1}^{1} \frac{e^{ikax} x}{\sqrt{1-x^2}} dx = \frac{1}{i} \frac{d}{d(ka)} \pi J_0(ka) = i\pi J_1(ka)
\]

The problem is that the evaluation of (2.33) is very time consuming because the k-integrand has fast oscillations (see Fig. 7). Nevertheless it seems that the contribution from the first term in the right hand side of (2.32) cannot be neglected when \( v \) is close to \( u \). For example, the \( v \)-integrand of \( E_v \) for \( v=22.61 \) gives 112.203 while the second term in the right hand side of (2.32) gives -26.577 (the range of integration in \( k \) is \( 0 \leq k \leq 113865 \)).
Fig. 8. $k$-integrand of (2.25) for $p=1$, $v=22.61$, $\theta = 0$

Fig. 9. $k$-integrand of (2.25) for $p=1$, $v=22.61$, $\theta = 0$

Fig. 10. $k$-integrand of (2.26) for $p=1$, $v=22.61$, $\theta = 0$. 
Fig. 11. k-integrand of (2.24) for p=1, v=22.61, θ = 0.0628

Fig. 12. k-integrand of (2.25) for p=1, v=22.61, θ = 0.0628

Fig. 13. k-integrand of (2.26) for p=1, v=22.61, θ = 0.0628
Fig. 14. $k$-integrand of (2.24) for $p=1$, $v=22.61$, $\theta = 0.628$

Fig. 15. $k$-integrand of (2.25) for $p=1$, $v=22.61$, $\theta = 0.628$

Fig. 16. $k$-integrand of (2.26) for $p=1$, $v=22.61$, $\theta = 0.628$
Fig. 17. $k$-integrand of (2.24) for $p=1$, $v=22.61$, $\theta = \frac{\pi}{2}$

Fig. 18. $k$-integrand of (2.25) for $p=1$, $v=22.61$, $\theta = \frac{\pi}{2}$

Fig. 19. $k$-integrand of (2.26) for $p=1$, $v=22.61$, $\theta = \frac{\pi}{2}$
Fig. 20. Left: $\theta$-integrand of (2.24) for $p=1$, $k=10$. Right: the integrand of (2.24) has been put in the form $kg(k, \theta) \exp(-\frac{k^2}{2}f(\theta))$. In red is plotted the term $k \exp(-\frac{k^2}{2}f(\theta))$, in blue the term $kg(k, \theta)$.

Fig. 21. Left: $\theta$-integrand of (2.24) for $p=1$, $k=50$. Right: the same as in 20.

Fig. 22. Left: $\theta$-integrand of (2.24) for $p=1$, $k=150$. Right: the same as in 20.
Fig. 23. Left: $\theta$-integrand of (2.24) for $p=1$, $k=300$. Right: the same as in 20

Fig. 24. Left: $\theta$-integrand of (2.24) for $p=1$, $k=600$. Right: the same as in 20

Fig. 25. Left: $\theta$-integrand of (2.24) for $p=1$, $k=2000$. Right: the same as in 20
Fig. 26. $\theta$-integrand of (2.24) for $p=1$, $k=5000$. Right: the same as in 20

Fig. 27. $\theta$-integrand of (2.24) for $p=1$, $k=10000$. Right: the same as in 20

Fig. 28. $\theta$-integrand of (2.24) for $p=1$, $k=20000$. Right: the same as in 20
Fig. 29. Left: $\theta$-integrand of (2.24) for $p=1, k=50000$. Right: the same as in 20

Fig. 30. Left: $\theta$-integrand of (2.24) for $p=1, k=100000$. Right: the same as in 20
Fig. 31. $k$-integrand of (2.25) for $p=1$, $v=22.41$, $\theta = 0$

Fig. 32. $k$-integrand of (2.25) for $p=5$, $v=22.41$, $\theta = 0$

Fig. 33. $k$-integrand of (2.25) for $p=9$, $v=22.41$, $\theta = 0$
Fig. 34. k-integrand of (2.25) for $p=1$, $v=22.41$, $\theta = \frac{\pi}{3}$

Fig. 35. k-integrand of (2.25) for $p=5$, $v=22.41$, $\theta = \frac{\pi}{3}$

Fig. 36. k-integrand of (2.25) for $p=1$, $v=22.41$, $\theta = \frac{\pi}{3}$
Fig. 37. $k$-integrand of (2.25) for $p=1, v=22.41, \theta = \frac{\pi}{2}$.

Fig. 38. $k$-integrand of (2.26) for $p=5, v=22.41, \theta = \frac{\pi}{2}$.

Fig. 39. $k$-integrand of (2.26) for $p=9, v=22.41, \theta = \frac{\pi}{2}$. 

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Fig. 40. $k$-integrand of (2.26) for $p=1$, $v=22.41$, $\theta = 0$.

Fig. 41. $k$-integrand of (2.26) for $p=5$, $v=22.41$, $\theta = 0$.

Fig. 42. $k$-integrand of (2.26) for $p=9$, $v=22.41$, $\theta = 0$. 
Fig. 43. k-integrand of (2.26) for $p=1$, $v=22.41$, $\theta = \frac{\pi}{3}$.

Fig. 44. k-integrand of (2.26) for $p=5$, $v=22.41$, $\theta = \frac{\pi}{3}$.

Fig. 45. k-integrand of (2.26) for $p=9$, $v=22.41$, $\theta = \frac{\pi}{3}$.
Fig. 46. $k$-integrand of (2.26) for $p=1$, $v=22.41$, $\theta = \frac{\pi}{2}$.

Fig. 47. $k$-integrand of (2.26) for $p=5$, $v=22.41$, $\theta = \frac{\pi}{2}$.

Fig. 48. $k$-integrand of (2.26) for $p=9$, $v=22.41$, $\theta = \frac{\pi}{2}$.
Fig. 49. $k$-integrand of (2.25) for $p=1$, $v=22.78$, $\theta = 0$

Fig. 50. $k$-integrand of (2.25) for $p=5$, $v=22.78$, $\theta = 0$

Fig. 51. $k$-integrand of (2.25) for $p=9$, $v=22.78$, $\theta = 0$
Fig. 52. k-integrand of (2.25) for \( v=22.78, \theta = \frac{\pi}{3} \). Left: \( p=1 \). Right: \( p=5 \).

Fig. 53. Left: k-integrand of (2.25) for \( p=9, v=22.78, \theta = \frac{\pi}{3} \). Right: k-integrand of (2.25) for \( p=1, v=22.78, \theta = \frac{\pi}{2} \).

Fig. 54. k-integrand of (2.25) for \( v=22.78, \theta = \frac{\pi}{2} \). Left: \( p=5 \). Right: \( p=9 \).
Fig. 55. $k$-integrand of (2.26) for $p=1$, $v=22.78$, $\theta = 0$.

Fig. 56. $k$-integrand of (2.26) for $p=5$, $v=22.78$, $\theta = 0$.

Fig. 57. $k$-integrand of (2.26) for $p=9$, $v=22.78$, $\theta = 0$. 

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Fig. 58. Fig. 58. $k$-integrand of (2.26) for $p=1$, $v=22.78$, $\theta = \frac{\pi}{3}$.

Fig. 59. Fig. 59. $k$-integrand of (2.26) for $p=5$, $v=22.78$, $\theta = \frac{\pi}{3}$.

Fig. 60. Fig. 60. $k$-integrand of (2.26) for $p=9$, $v=22.78$, $\theta = \frac{\pi}{3}$.
Fig. 61. Fig. 61. k-integrand of (2.26) for $p=1$, $v=22.78$, $\theta = \frac{\pi}{2}$.

Fig. 62. Fig. 62. k-integrand of (2.26) for $p=5$, $v=22.78$, $\theta = \frac{\pi}{2}$.

Fig. 63. Fig. 63. k-integrand of (2.26) for $p=9$, $v=22.78$, $\theta = \frac{\pi}{2}$. 
Fig. 64. Fig. 64. $k$-integrand of (2.25) for $p=1$, $v=22.88$, $\theta = 0$.

Fig. 65. Fig. 65. $k$-integrand of (2.25) for $p=5$, $v=22.88$, $\theta = 0$.

Fig. 66. Fig. 66. $k$-integrand of (2.25) for $p=9$, $v=22.88$, $\theta = 0$. 
Fig. 67. $k$-integrand of (2.25) for $v=22.88$, $\theta = \frac{\pi}{3}$. Left: $p=1$. Right: $p=5$.

Fig. 68. $k$-integrand of (2.25) $v=22.88$. Left: $p=9$, $\theta = \frac{\pi}{3}$. Right: $p=1$, $\theta = \frac{\pi}{2}$.

Fig. 69. $k$-integrand of (2.25) for $v=22.88$, $\theta = \frac{\pi}{2}$. Left: $p=5$. Right: $p=9$.
Fig. 70. $k$-integrand of (2.26) for $p=1$, $v=22.88$, $\theta = 0$.

Fig. 71. $k$-integrand of (2.26) for $p=5$, $v=22.88$, $\theta = 0$.

Fig. 72. $k$-integrand of (2.26) for $p=9$, $v=22.88$, $\theta = 0$. 
Fig. 73. k-integrand of (2.26) for \( v=22.88, \theta = \frac{\pi}{3} \). Left: \( p=1 \). Right: \( p=5 \)

Fig. 74. k-integrand of (2.26) for \( v=22.88 \). Left: \( p=9, \theta = \frac{\pi}{3} \). Right: \( p=1, \theta = \frac{\pi}{2} \).

Fig. 75. k-integrand of (2.26) for \( v=22.88, \theta = \frac{\pi}{2} \). Left: \( p=5 \). Right: \( p=9 \).
Fig. 76. $\theta$-integrand of (2.25) for $v=22.41, k=2000$. Left: $p=1$. Right: $p=5$

Fig. 77. Left: $\theta$-integrand of (2.25) for $p=9, v=22.78, k=2000$. Right: $\theta$-integrand of (2.25) for $p=1, v=22.78, k=2000$.

Fig. 78. $\theta$-integrand of (2.25) for $v=22.78, k=2000$. Left: $p=5$. Right: $p=9$
Fig. 79. Fig. 79. $\theta$-integrand of (2.25) for $v=22.88$, $k=2000$. Left: $p=1$. Right: $p=5$

Fig. 80. Fig. 80. Left: $\theta$-integrand of (2.25) for $p=9$, $v=22.88$, $k=2000$. Right: $\theta$-integrand of (2.24) for $p=1$, $v=22.78$, $k=130$.

Fig. 81. Fig. 81. $\theta$-integrand of (2.24) for $v=22.78$, $k=130$. Left: $p=5$. Right: $p=9$
Fig. 82. \( \theta \)-integrand of (2.24) for \( v=22.88, k=462 \). Left: \( p=1 \). Right: \( p=5 \).

Fig. 83. Left: \( \theta \)-integrand of (2.24) for \( p=5, v=22.88, k=462 \). Right: \( \theta \)-integrand of (2.24) for \( p=1, v=22.41, k=33 \).

Fig. 84. \( \theta \)-integrand of (2.24) for \( v=22.41, k=33 \). Left: \( p=5 \). Right: \( p=9 \).