A Numerical Study of the Vlasov-Fokker-Planck Equation with Applications to Particle Beam Dynamics

by

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B.S., Mathematics, University of New Mexico, 2003

THESIS

Submitted in Partial Fulfillment of the Requirements for the Degree of

Master of Science
Mathematics

The University of New Mexico
Albuquerque, New Mexico

July, 2005
Dedication

To my wife and my mother.
Acknowledgments

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ABSTRACT OF THESIS

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Abstract

The motion of particles in a bunched beam traveling in a particle accelerator damping ring is described by a distribution function satisfying a nonlinear partial differential integral equation known as the Vlasov-Fokker-Planck (VFP) equation.

The VFP equation contains a term resulting from the collective force arising from the inter-particle interaction within the bunch. In this thesis, the collective force is taken to be a sum of two terms; a term due to the interaction of the bunch with the beam pipe and another term due to the coherent synchrotron radiation emitted by particles within the bunch.
The main objectives of this thesis are to describe a numerical algorithm to solve the VFP equation in the space-time domain, to apply the algorithm to two important accelerators, and to interpret the results obtained.
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Chapter 1

Statement of the Problem

1.1 Description of a Damping Ring

A modern particle accelerator is a massive structure which accelerates bunched particle beams to velocities near the speed of light and then collides the beams head on. Its purpose is to give physicists an opportunity to study the basic properties of energy and matter. Particle accelerators can be either circular or linear in design. In some experiments, one beam consists of electrons $e^-$ and the other beam consists of positrons $e^+$. This thesis will focus on electron and positron beams. In this chapter, we study the particle dynamics in the damping rings at the Stanford Linear Accelerator. Then in Chapter 4, we take a look at a different machine known as the VUV Storage Ring located at Brookhaven National Laboratory. First, we discuss what a linear collider is.

A linear collider is one type of a modern particle accelerator. They are used for experiments where the particle beams have high energies. In a linear collider the particle beams are accelerated along a straight line in linacs. This reduces the amount of synchrotron radiation emitted by the beams in comparison to their circular sisters.
Chapter 1. Statement of the Problem

Synchrotron radiation has the detrimental effect of decreasing the beam energy $E$, but is negligible in linear colliders even for high energies. Therefore, if one wishes to conduct experiments with high energy beams, then a linear collider is best suited for these types of experiments. After being accelerated along a straight line, the particle beams are brought into collision at the end of the beam line.

Since the density of a bunch is very small compared to solid matter, there is only a small interaction probability in two colliding bunches. Therefore, a great deal of effort must be made to obtain a high event rate during experiments. The number of events per second is given by the differential equation $\dot{N}_p = \sigma_p \cdot \mathcal{L}$, where $\sigma_p$ is the cross-sectional “size” or the probability of the desired event and $\mathcal{L} = (f_{\text{rev}} N_1 N_2)/(4\pi \sigma_x \sigma_y)$ is a measure of the interaction probability in the colliding beams. The factor $\mathcal{L}$ is called the luminosity and determines the performance of the particle accelerator. In this expression for the luminosity, $N_1$ and $N_2$ are the number of particles in the two colliding beams, $\sigma_x$ and $\sigma_y$ are the respective horizontal and vertical sizes of the bunches in the collision region, and $f_{\text{rev}}$ is the frequency of beam crossings. It is clear that the luminosity can be increased by increasing $N_1$, or $N_2$, or both. The luminosity can also be increased by decreasing the horizontal and vertical sizes of the bunched beam in the collision region. In a linear collider, such as the Stanford Linear Accelerator (SLAC), the frequency of the beam crossings is very low compared to circular accelerators. Also, the number of particles in the beams cannot be made too large, because it is very difficult to stably bring high current beams from zero energy up to a final energy. Thus, if one wants to increase the luminosity in a linear collider, it is essential to reduce the cross-section of the bunches. This is done by injecting the beams into devices known as damping rings before they are sent into the linear collider. For more information on damping rings and the physics of particular accelerators, the reader is referred to [9].

At SLAC there are two damping rings located on opposite sides of the accelerator.
Electron beams are sent to the North Damping Ring, while positron beams are sent to the South Damping Ring. The main purpose of the SLAC damping rings is to decrease the cross-sectional spread of the particle beams while maintaining a constant energy. As the particles circulate in the two damping rings, they lose energy due to the emission of incoherent synchrotron radiation (ISR) by the individual particles. To counter this loss of energy, the particles are re-accelerated each time they pass through an r.f. cavity which creates an electric field. In summary, the ISR has a tendency to decrease the transverse motion of a particle bunch, while the cavity re-accelerates the beam in the desired direction of motion.

1.2 Physical and Mathematical Description

Following the outline presented in [2], particle motion in a damping ring is governed by the Hamiltonian

\[ H(z, t, f(\cdot, t)) = H_e(z, t) + H_c(z, t, f(\cdot, t)), \]  

(1.1)

where \( f(z, t) \) is the phase space particle density, \( z = (q, p) \) is a 2\( d \)-dimensional phase space variable, and \( d \) is the number of degrees of freedom. In mathematical terms, \( f(z, t) \) can be thought of as a probability distribution function. In the expression for the Hamiltonian, the first term \( H_e(z, t) \) accounts for all the external fields on a single particle, and the second term \( H_c(z, t, f(\cdot, t)) \) accounts for the sum of all forces on a single particle due to all the other particles within the bunch. (When we write \( f(\cdot, t) \) as the argument of a function, it means that the function depends on \( f(\xi, t) \) for all \( \xi \)).

If the particle motion starting at \( z_0 \) at time \( t_0 \) evolves to \( z \) at time \( t \), then particle conservation (conservation of probability) requires that

\[ f(z, t) \, dz = f(z_0, 0) \, dz_0. \]  

(1.2)
Chapter 1. Statement of the Problem

If the system is Hamiltonian, as is our case, then the time evolution of the system is volume-preserving, i.e. \( dz = dz_0 \). Therefore, (1.2) implies that \( f(z(t), t) \) does not explicitly depend on \( t \). Equivalently, this can be expressed by saying

\[
\frac{d}{dt} f(z, t) = \frac{\partial f}{\partial t} + \frac{\partial f}{\partial q} \frac{dq}{dt} + \frac{\partial f}{\partial p} \frac{dp}{dt} = 0. \tag{1.3}
\]

Applying Hamilton’s equations of motion, (1.3) becomes Vlasov’s integro-partial-differential equation

\[
\frac{d}{dt} f(z, t) = \frac{\partial f}{\partial t} + \frac{\partial f}{\partial q} \frac{\partial H}{\partial p} - \frac{\partial f}{\partial p} \frac{\partial H}{\partial q} = 0. \tag{1.4}
\]

In the most general classical setting, the collective term \( H_c(z, t, f(\cdot, t)) \) somehow comes from the Lorentz force \( F_c(z, t, f(\cdot, t)) = e(\mathbf{E} + v \times \mathbf{B}) \), where \( e \) is the electron charge, and \( \mathbf{E} \) and \( \mathbf{B} \) are the electric and magnetic fields coming from a charge density \( \rho \) and a current density \( J \) [10]. The two fields \( \mathbf{E} \) and \( \mathbf{B} \) are derived from Maxwell’s equations

\[
\nabla \cdot \mathbf{E} = \frac{1}{\epsilon_0} \rho, \tag{1.5}
\]

\[
\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}, \tag{1.6}
\]

\[
\nabla \cdot \mathbf{B} = 0, \tag{1.7}
\]

\[
\nabla \times \mathbf{B} = \mu_0 J + \mu_0 \epsilon_0 \frac{\partial \mathbf{E}}{\partial t}. \tag{1.8}
\]

In general, the coupled Vlasov-Maxwell system of equations is much too difficult to solve and approximations need to be made. Such an approximation is discussed in the next section which will include the effect of the beam pipe and coherent synchrotron radiation.

Coherent synchrotron radiation, or CSR, is the coherent radiation arising from the electron bunching along the circular trajectory [11],[12]. If a single electron of total energy \( E \) and rest energy \( E_r \) moves in a circle of radius \( R \), then it radiates
energy at a rate $L \ (eV \ per \ turn)$, given by:

$$L = 400\pi \frac{e}{R} \left( \frac{E}{E_r} \right)^4,$$  \hspace{1cm} (1.9)

where $e$ is the electron charge and $E \gg E_r$. In a synchrotron, a concentrated bunch of electrons moves in a circular orbit, and the total amount of radiation depends on the coherence between the waves from the ISR emitted by the individual electrons. If there were complete coherence, the radiation per electron would be $N \cdot L$, where $N$ is the bunch population. CSR gives a radiated power comparable to $N^2$ \cite{4}, and is independent of $E$. This radiated energy loss is much larger that the energy loss due to ISR emitted by the individual particles. If $N$ were large, the CSR could actually prevent the operation of the synchrotron if it were not shielded by the beam pipe.

An important quantum effect to include is the damping and diffusion due to the incoherent emission of photons in ISR. The damping and diffusion are described as increments of a random (Wiener) process; the momentum $p$ receives random kicks that have a Gaussian distribution \cite{2}. Therefore, the right hand side of (1.4) must be adjusted to include Fokker-Planck terms. Consequently, the Vlasov equation is transformed into the Vlasov-Fokker-Planck equation

$$\frac{\partial f}{\partial t} + \frac{\partial f}{\partial q} \frac{\partial H}{\partial p} - \frac{\partial f}{\partial p} \frac{\partial H}{\partial q} = D \left[ \frac{\partial (pf)}{\partial p} + \frac{\partial^2 f}{\partial p^2} \right],$$ \hspace{1cm} (1.10)

where $D = 2\beta$ is the diffusion constant of the random process, and $\beta$ is the damping rate.

### 1.3 Longitudinal Motion with CSR and Wake Potential

We consider longitudinal motion of particles in the SLAC damping rings with CSR and a machine wake potential. As stated earlier, the CSR is the coherent radiation
emitted by the particles in a bunch. On the other hand, the wake potential is a consequence of the particle interaction with the various metallic structures of the beam pipe. It essentially comes from solving Maxwell’s equations with boundary conditions. The CSR and wake potential affect the dynamics of the system, and both are included in the collective force.

A damping ring is designed with a synchronous particle trajectory and a nominal energy kept in mind. The idea is that the synchronous particle trajectory will close on itself after one revolution. If a particle with nominal energy $E_0$ and the right phase with respect to the r.f. cavity is put on this path, the particle will continue to stay on this path for all time without losing or gaining energy on average. A particle traveling along this path is called the *synchronous particle*, and general particle motion is referenced to the synchronous particle. This motion is in 3 degrees of freedom. However, to good approximation the transverse and longitudinal degrees of freedom are uncoupled. For this thesis we focus on the longitudinal degree of freedom. For more information on our basic coordinate system, the reader can consult [1].

We will describe the longitudinal particle motion using normalized, dimensionless independent variables:

$$
q = \frac{z}{\sigma_z}, \quad p = -\text{sgn}(\eta) \frac{E - E_0}{\sigma_E}, \quad \tau = \omega_s t .
$$

(1.11)

Here $z = s - s_0$ is the distance (in arc length along the trajectory) to the synchronous particle, $\eta$ is known as the slip factor, $E$ is the particle energy, $\sigma_z$ is the rms bunch length, $\sigma_E$ is the rms energy spread, $t$ is the time variable, and $\omega_s$ is known as the synchronous frequency. By convention, $z > 0$ when the test particle leads the synchronous particle. A list of the most important parameters and quantities for the SLAC damping rings is given in table 1.1 with an r.f. voltage of 800 keV. A complete list of parameters can be found in Podobedov’s thesis [1]. The bend radius $R$ is defined to be the radius of curvature of the bending magnets located at the four corners of the damping rings and should not be confused with the radius of
The damping ring itself. Also, the values for the energy damping time and diffusion constant are for the South Damping Ring only.

For this study, the longitudinal motion due to the r.f. cavity is described by the Hamiltonian for a harmonic oscillator,

\[ H_e(q, p) = \frac{q^2 + p^2}{2}. \]  

(1.12)

The equations of motion due to the r.f. cavity become

\[ \frac{dq}{d\tau} = p, \quad \frac{dp}{d\tau} = -q, \quad \tau = \omega_s t. \]  

(1.13)

The collective term \( H_c \) in the Hamiltonian is related to the collective force through the integral

\[ H_c(q, f(\cdot, \tau)) = -I \int_q^\infty F(q', f(\cdot, \tau)) \, dq', \]  

(1.14)

where \( I \) is the current parameter measured in pC/V, and \( F(q, f(\cdot, \tau)) \) is the collective force measured in V/pC. The current parameter \( I \) is given by

\[ I = \frac{Ne^2}{2\pi \nu_s \sigma_E}, \]  

(1.15)

<table>
<thead>
<tr>
<th>Quantity</th>
<th>Symbol</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>Nominal Energy</td>
<td>( E_0 )</td>
<td>( 1.19 \times 10^9 ) eV = 1.19 GeV</td>
</tr>
<tr>
<td>Synchrotron Frequency</td>
<td>( \omega_s = 2\pi f_s )</td>
<td>6.18 \times 10^9 Hz = 0.618 MHz</td>
</tr>
<tr>
<td>Revolution Frequency</td>
<td>( f_0 )</td>
<td>8.5 \times 10^6 Hz = 8.5 MHz</td>
</tr>
<tr>
<td>Bend Radius</td>
<td>( R )</td>
<td>2.037 m</td>
</tr>
<tr>
<td>Momentum Compaction</td>
<td>( \alpha )</td>
<td>0.015</td>
</tr>
<tr>
<td>Synchrotron Tune</td>
<td>( \nu_s = f_s/f_0 )</td>
<td>0.012</td>
</tr>
<tr>
<td>Lorentz Factor</td>
<td>( \gamma_0 = E_0/(mc^2) )</td>
<td>2330</td>
</tr>
<tr>
<td></td>
<td>( \beta_0 = \sqrt{1 - 1/\gamma_0^2} )</td>
<td>1</td>
</tr>
<tr>
<td></td>
<td>( \eta = \alpha - 1/\gamma_0^2 )</td>
<td>0.015</td>
</tr>
<tr>
<td>Slip Factor</td>
<td>( \tau_d )</td>
<td>1.78 ms = 1.78 \times 10^{-3} ) s</td>
</tr>
<tr>
<td>Energy Damping Time</td>
<td>( D = 2/\omega_s \tau_d )</td>
<td>0.00182</td>
</tr>
<tr>
<td>Diffusion Constant</td>
<td>( D )</td>
<td>0.00182</td>
</tr>
</tbody>
</table>

Table 1.1: fixed parameters taken from Podobedov’s thesis.
Chapter 1. Statement of the Problem

where \(N\) is the number of particles in the beam and \(e\) is the electron charge. Note that \(I\) is not really a current and \(F\) is not really a force, because they do not have the correct units of measure. However, \(I\) is proportional to \(N\), which is related to the current, and \(IF\) acts like a forcing term in the equations of motion.

It turns out that \(\sigma_z\) and \(\sigma_E\) are related by the equation

\[
\frac{\beta_0 \omega_s \sigma_z}{c} = \left| \eta \right| \frac{\sigma_E}{E_0}. \tag{1.16}
\]

For the SLAC damping ring, we choose the value \(\sigma_z = 0.004946\, \text{m}\) for the rms bunch length. This implies that the value for \(\sigma_E\) is 0.81 MeV, which is now a scaling constant to define the dimensionless variable \(p\), and not the rms energy spread.

With \(H_e\) and \(H_c\) defined, the Vlasov-Fokker-Planck equation now becomes

\[
\frac{\partial f}{\partial \tau} + p \frac{\partial f}{\partial q} - [q + IF(q, f(\cdot, \tau))] \frac{\partial f}{\partial p} = D \left[ \frac{\partial (pf)}{\partial p} + \frac{\partial^2 f}{\partial p^2} \right], \tag{1.17}
\]

where the collective force can be written as \(F(q, f(\cdot, \tau)) = F_1(q, f(\cdot, \tau)) + F_2(q, f(\cdot, \tau))\).

The first term in the collective force is due to the CSR, and is defined by

\[
F_1(q, f(\cdot, \tau)) = -\omega_0 \sum_n \exp \left( in^s \sigma_z q R \right) Z(n, n \omega_0) \lambda_n \left( \frac{\tau}{\omega_s} \right). \tag{1.18}
\]

In this definition, \(Z(n, n \omega_0)\) is the CSR longitudinal impedance measured in ohms \(\Omega\), \(\omega_0 = \beta_0 c/R = 147\, \text{MHz}\), and \(\sigma_z\) is the low current rms bunch length. The quantity \(\lambda\) is called the line density of the beam, and \(\lambda_n\) is its Fourier transform. Both quantities are given by the pair of integrals

\[
\lambda \left( \theta, \frac{\tau}{\omega_s} \right) = \frac{R}{\sigma_z} \int_{-\infty}^{\infty} f \left( \frac{R}{\sigma_z} \theta, p, \tau \right) dp. \tag{1.19}
\]

\[
\lambda_n \left( \frac{\tau}{\omega_s} \right) = \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{-in\theta} \lambda \left( \theta, \frac{\tau}{\omega_s} \right) d\theta. \tag{1.20}
\]

The longitudinal impedance \(Z(n, n \omega_0)\) is called the elementary impedance which corresponds to modes with phase velocity equal to the particle velocity. The complete
impedance is denoted by $Z(n, \omega)$, and is a function of $n$ and of all frequencies $\omega$, not just the integer multiples of $\omega_0$. Following the discussion in [5], the complete impedance is given by

$$Z(n, \omega) = \frac{Z_0(\pi R)^2}{\beta_0 h} \sum_{p=1,3,5,\ldots} \Lambda_p \left[ \frac{\omega \beta_0}{c} J'_n(\gamma_p R) H_n^{(1)}(\gamma_p R) \right] + \left( \frac{\alpha_p}{\gamma_p} \right)^2 \frac{n}{R} J_n(\gamma_p R) H_n^{(1)}(\gamma_p R).$$

(1.21)

Here, $H_n^{(1)} = J_n + iY_n$, where $J_n$ and $Y_n$ are Bessel functions of the first and second kind respectively. The other quantities in this expression are $\alpha_p = \pi p/h$, $\gamma_p^2 = (\omega/c)^2 - \alpha_p^2$, $\Lambda_p = 2(\sin x/x)^2$, and $x = \alpha_p \delta h/2$. Here $h$ is the distance between the parallel plates, and $Z_0 = 120\pi \Omega$. We follow convention and define $Z_n = Z(n, n\omega_0)$. The impedance has the property $Z_n = Z_{-n}^*$, where $*$ denotes complex conjugation.

It should also be noted that the particle density on configuration space is defined as

$$\rho(q, \tau) = \int_{-\infty}^{\infty} f(q, p, \tau) \, dp.$$  

(1.22)

Using this expression for $\rho$ and some elementary calculus, we find a relationship between $\lambda$ and $\rho$:

$$\lambda \left( \theta, \frac{\tau}{\omega_s} \right) = \lambda(\theta, t) = \frac{R}{\sigma_z} \rho(q, \tau).$$  

(1.23)

Other quantities that will be of some interest later on are total charge, dimensionless bunch length and dimensionless energy spread ($Q$, $\sigma_q$, $\sigma_p$ respectively). These three quantities are defined by the following integrals

$$Q = \int_{-\infty}^{\infty} \rho(q, \tau) \, dq,$$  

(1.24)

$$\sigma_q^2 = \int_{-\infty}^{\infty} \rho(q, \tau)(q - \bar{q})^2 \, dq,$$  

(1.25)

$$\sigma_p^2 = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(q, p, \tau)(p - \bar{p})^2 \, dp \, dq.$$  

(1.26)
Chapter 1. Statement of the Problem

Here, \( \bar{q} \) and \( \bar{p} \) are the mean values of \( q \) and \( p \) respectively, and are given as

\[
\bar{q} = \int_{-\infty}^{\infty} \rho(q) q \, dq ,
\]

(1.27)

\[
\bar{p} = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} p f(p, q, \tau) \, dp \, dq .
\]

(1.28)

The quantities \( \sigma_q \) and \( \sigma_p \) will prove to be useful for determining the onset of instability that occurs as the current parameter \( I \) increases.

Next, the second term in the collective force is due to the machine wakefield and is defined as

\[
F_2(q, f(\cdot, \tau)) = \int_{-\infty}^{\infty} W_2(q - q') \left[ \int_{-\infty}^{\infty} f(q', p, \tau) \, dp \right] \, dq',
\]

(1.29)

where the wakefield \( W_2(q - q') \) gives the longitudinal electric field (averaged over one turn) on a test particle at \( q \) due to a point source at \( q' \) and has units of V/pC [2]. It is expressed as a potential difference over one turn with a positive value corresponding to a gain in energy. Causality requires that the wakefield be zero in front of the ultrarelativistic beam, i.e. \( W_2(q) = 0 \) for \( q > 0 \). Using (1.22), we can rewrite (1.29) as

\[
F_2(q, f(\cdot, \tau)) = \int_{-\infty}^{\infty} W_2(q - q') \rho(q', \tau) \, dq'.
\]

(1.30)

The Vlasov-Fokker-Planck equation has an equilibrium solution of the form

\[
f(q, p) = \frac{1}{\sqrt{2\pi}} e^{-p^2/2} \rho(q).
\]

(1.31)

This clearly satisfies the right-hand side of (1.17), and the left-hand side is satisfied if \( \rho(q) \) satisfies Haïssinski’s nonlinear integral equation [3]

\[
\rho(q) = \frac{\exp[-q^2/2 + I \int_q^{\infty} F(q', \rho) \, dq']}{\int_{-\infty}^{\infty} \exp[-q^2/2 + I \int_q^{\infty} F(q', \rho) \, dq']dq}. \tag{1.32}
\]

This is a nonlinear fixed point problem \( \rho = A(\rho) \). In an appropriate function space, it can be shown that for sufficiently small current \( I \), this fixed point problem has a unique solution.
Chapter 1. Statement of the Problem

For \( I = 0 \), \( \rho(q) \) is Gaussian, and for CSR alone, the solution of (1.32) stays roughly Gaussian for the currents of this thesis. Thus, for CSR alone, we take

\[
f(q, p) = \frac{1}{2\pi} e^{-(q^2+p^2)/2}.
\]

In order to calculate the true equilibrium, one would need to write the CSR impedance \( Z_n \) as a wake potential \( W_1 \) due to CSR. The relationship between an impedance and a wake potential is given by a pair of Fourier transforms [1]

\[
Z(\omega) = \frac{\sigma_z}{c} \int_{-\infty}^{\infty} W(q) e^{-i\omega q \sigma_z/c} dq
\]

\[
W(q) = \frac{1}{2\pi} \int_{-\infty}^{\infty} Z(\omega) e^{i\omega q \sigma_z/c} d\omega.
\]

However, care must be taken when using this approach, because the CSR wakefield \( W_1 \) is a rapidly varying function in a region small compared to typical mesh cells used in the numerical integration of equation (1.17). Consequently, we prefer to not use the equilibrium given this way. Rather, we prefer to use an approximate equilibrium solution defined by (1.33).
Chapter 2

Numerical Algorithm

2.1 Numerical Algorithm for the Solving the VFP Equation

In this chapter, we describe a numerical algorithm [2] for solving equation (1.17) over one time step from $\tau_0$ to $\tau_0 + \Delta \tau$. Throughout our discussion, we assume the collective force contains both terms from the CSR and machine wakefield. We invoke an operator splitting technique and treat the Vlasov and Fokker-Planck operators separately.

We first consider the propagation of $f(q, p, \tau)$ by the Vlasov operator. Here, we lay down an algorithm to solve Vlasov’s equation without the Fokker-Planck terms:

\[
\frac{\partial f}{\partial \tau} + p \frac{\partial f}{\partial q} - [q + IF(q, f(\cdot, \tau))] \frac{\partial f}{\partial p} = 0,
\]

where $F(q, f(\cdot, \tau))$ is the collective force due to the CSR impedance and the machine wakefield. We set up a uniformly spaced square grid to represent the domain of the
distribution function:

\[ q_j = j \frac{\kappa}{N}, \quad p_k = k \frac{\kappa}{N}, \quad j, k = 0, 1, \ldots, 2N. \]  

(2.2)

Here, \( \kappa \) defines the boundary of the square grid and \( N \) defines the grid resolution. The charge density \( \rho(q, \tau) \) is assumed to be zero outside the interval \([ -\kappa, \kappa ]\). Typically, \( \kappa \) is between six and seven and \( N \) is between 200 and 600. If \( N = 200 \), we have a \( 401 \times 401 \) grid. Now suppose we know the solution \( f(q, p, \tau) \) at the grid points \((q_j, p_k)\) and at time \( \tau = \tau_0 \). Our goal is to find the solution \( f(q, p, \tau) \) at these same grid points but at a new time \( \tau = \tau_0 + \Delta \tau \).

Next, we freeze the phase space density \( f(q_j, f(\cdot, \tau)) \) in the collective force \( F(q_j, f(\cdot, \tau)) \) at \( \tau = \tau_0 \). Consequently, this freezes the collective force itself, i.e.

\[ F(q_j, f(\cdot, \tau)) = F(q_j, f(\cdot, \tau_0)) = F_1(q_j, f(\cdot, \tau_0)) + F_2(q_j, f(\cdot, \tau_0)), \]  

(2.3)

for \( \tau \in [\tau_0, \tau_0 + \Delta \tau] \). In particular, this says that \( F(q_j, f(\cdot, \tau_0 + \Delta \tau)) = F(q_j, f(\cdot, \tau_0)) \).

The force due to the machine wakefield (1.30) is approximated by Simpson’s rule

\[ F_2(q_j, f(\cdot, \tau_0 + \Delta \tau)) = F_2(q, f(\cdot, \tau_0)) \approx \sum_{k=j}^{2N} W(q_j - q_k) \rho(q_k, \tau_0) w_k, \]  

(2.4)

where \( w_k \) are the weights from Simpson’s integration of (1.30). An algorithm for evaluating the collective force due to CSR (1.18) is given in the next section.

Now if we define a new quantity \( H(q, \tau_0) \) by the expression

\[ H(q, \tau_0) = q + IF(q, f(\cdot, \tau_0)), \]  

(2.5)

then we have the following Liouville initial value problem:

\[ \frac{\partial f}{\partial \tau} + p \frac{\partial f}{\partial q} - H(q, \tau_0) \frac{\partial f}{\partial p} = 0, \]  

(2.6)

for \( \tau \in [\tau_0, \tau_0 + \Delta \tau] \), and with \( f(q, p, \tau_0) \) given at \( q = q_j, p = p_k \). This leads to the
Chapter 2. Numerical Algorithm

characteristic differential equations:
\[
\frac{dq}{d\tau} = p, \quad q(\tau_0) = q_0 \tag{2.7}
\]
\[
\frac{dp}{d\tau} = -H(q, \tau_0), \quad p(\tau_0) = p_0. \tag{2.8}
\]

This system of IVP’s can also be written as
\[
\frac{dz}{d\tau} = G(z) = \begin{bmatrix}
G_1(z) \\
G_2(z)
\end{bmatrix}, \quad z(\tau_0) = z_0, \tag{2.9}
\]
where \(G_1(z) = p, \ G_2(z) = -H(q, \tau_0), \) and \(z_0 = (q_0, p_0)\). Let \(z(\tau) = \Phi(\tau, \tau_0, z_0)\) be the solution to (2.9). Now we state the main theorem of the thesis.

**Theorem 1.** \(f(z, \tau) = f(\Phi(\tau_0, \tau, z), \tau_0)\) for \(\tau \in [\tau_0, \tau_0 + \Delta \tau]\).

**Proof.** Let \(\dot{z} = \Phi(\tau_0, \tau, z)\) then \(z = \Phi(\tau, \tau_0, \dot{z})\) by the Uniqueness theorem from ODE’s. Now the theorem can be stated as \(f(\Phi(\tau, \tau_0, \dot{z}), \tau_0) = f(\dot{z}, \tau_0)\).

a) First, it’s obvious that equality holds when \(\tau = \tau_0:\)
\[
f(\Phi(\tau_0, \tau_0, \dot{z}), \tau_0) = f(\dot{z}, \tau_0). \tag{2.10}
\]

b) Now let’s differentiate \(f(\Phi(\tau, \tau_0, \dot{z}), \tau)\) with respect to \(\tau,
\[
D_1f(\Phi(\tau, \tau_0, \dot{z}), \tau) \frac{\partial \Phi(\tau, \tau_0, \dot{z})}{\partial \tau} + D_2f(\Phi(\tau, \tau_0, \dot{z}), \tau) =
\]
\[
D_1f(\Phi(\tau, \tau_0, \dot{z}), \tau) G(\Phi(\tau, \tau_0, \dot{z})) + D_2f(\Phi(\tau, \tau_0, \dot{z}), \tau). \tag{2.11}
\]

However, this equals zero because (2.6) can be written as \(D_1f(z, \tau)G(z) + D_2f(z, \tau) = 0\). Thus, the theorem holds. \(\square\)

In particular, the theorem gives us \(f(q_j, p_k, \tau_0 + \Delta \tau_0) = f(\dot{q}_j, \dot{p}_k, \tau_0)\), where \(\dot{z}_{jk} = (\dot{q}_j, \dot{p}_k) = \Phi(\tau_0, \tau_0 + \Delta \tau, z_{jk})\) and \(z_{jk} = (q_j, p_k)\). It’s almost surely the case that \(\dot{z}_{jk}\) is not on a grid point. Therefore, we have to evaluate \(f(\dot{q}_j, \dot{p}_k, \tau_0)\) using some
interpolation scheme (biquadratic interpolation seems adequate). However, we first need to find \( \hat{z}_{jk} \).

We rewrite the solution to (2.9) as
\[
z(\tau) = \Phi(\tau, \tau_0, z_0) = M(\tau|\tau_0)z_0,
\]
where \( M(\tau|\tau_0) \) is a nonlinear map defining the dynamics of the system. The notation \( z_1 = M(\tau|\tau_0)z_0 \) means that the particle passes through \( z_0 \) at time \( \tau_0 \) and through \( z_1 \) at time \( \tau \). One could use Euler’s method to integrate \( \Phi(\tau, \tau_0, z_0) \) backwards from \( \tau = \tau_0 + \Delta \tau \) to \( \tau = \tau_0 \) to find \( \hat{z}_{jk} \). However, Euler’s method pays too little attention to the conservation of probability (1.2). It has been discovered that for a small step \( \Delta \tau \) the map \( M(\tau_0 + \Delta \tau|\tau_0) \) can be represented approximately by a composition of a rotation map \( R \) through an angle \( \Delta \tau \) and a kick \( K \) of \( p \) at fixed \( q \); \( M(\tau + \Delta \tau|\tau) = K \circ R \), where \( K \) and \( R \) are symplectic (area preserving) maps.

Hence,
\[
\hat{z}_{jk} = M(\tau|\tau + \Delta \tau) \approx R^{-1} \circ K^{-1}(z_{jk})
\]
\[
= \begin{pmatrix}
\cos \Delta \tau & -\sin \Delta \tau \\
\sin \Delta \tau & \cos \Delta \tau
\end{pmatrix}
\begin{pmatrix}
q_j \\
p_k + IF(q_j, f(\cdot, \tau_0))\Delta \tau
\end{pmatrix}
\]
\[
= \begin{pmatrix}
q_j \cos \Delta \tau - \sin \Delta \tau[p_k + IF(q_j, f(\cdot, \tau_0))\Delta \tau] \\
q_j \sin \Delta \tau + \cos \Delta \tau[p_k + IF(q_j, f(\cdot, \tau_0))\Delta \tau]
\end{pmatrix}
\]
\[
= \begin{pmatrix}
\hat{q}_j \\
\hat{p}_k
\end{pmatrix}
\]

With \( \hat{z}_{jk} \) found, our next task is evaluating \( f(\hat{q}_j, \hat{p}_k, \tau_0) \) using biquadratic interpolation. First, we find the grid cell \((s, t)\) in which \( z_{jk} \) is located by defining the integers \( s \) and \( t \) as
\[
s = \text{int} \left( \frac{\hat{q}_j}{\Delta} \right), \quad t = \text{int} \left( \frac{\hat{p}_k}{\Delta} \right)
\]
Here, $\Delta = \Delta q = \Delta p$ is the grid step size and \textit{int} is used to denote the integer part of the quantity. Next, we define the quantities $x$ and $y$ by the equations

$$x = \frac{\hat{q}_j - q_s}{\Delta}, \quad y = \frac{\hat{p}_k - p_t}{\Delta} \quad (2.18)$$

Now we are in position to apply the nine point biquadratic formula:

$$4f(q_j, p_k, \tau_0 + \Delta \tau) =$$

$$x(x - 1)[y(y - 1)f(q_{s-1}, p_{t-1}, \tau_0) + 2(1 - y^2)f(q_{s-1}, p_t, \tau_0) + y(y + 1)f(q_{s-1}, p_{t+1}, \tau_0)] +$$

$$2(1 - x^2)[y(y - 1)f(q_{s-1}, p_{t-1}, \tau_0) + 2(1 - y^2)f(q_{s-1}, p_t, \tau_0) + y(y + 1)f(q_{s-1}, p_{t+1}, \tau_0)] +$$

$$x(x + 1)[y(y - 1)f(q_{s-1}, p_{t-1}, \tau_0) + 2(1 - y^2)f(q_{s-1}, p_t, \tau_0) + y(y + 1)f(q_{s-1}, p_{t+1}, \tau_0)].$$

With this formula, we have found the solution to Vlasov’s equation (2.6) for one time step.

Next, we consider the propagation by the Fokker-Planck operator, and integrate the Vlasov-Fokker-Planck equation

$$\frac{\partial f}{\partial \tau} + \frac{\partial f}{\partial q} - [q + IF(q, f(\cdot, \tau))] \frac{\partial f}{\partial p} = D \left[ \frac{\partial(pf)}{\partial p} + \frac{\partial^2 f}{\partial p^2} \right]. \quad (2.19)$$

We approximate the right-hand side by a finite difference method

$$D \left[ \frac{\partial(pf)}{\partial p} + \frac{\partial^2 f}{\partial p^2} \right] \approx \frac{D}{\Delta p} [G_{j,k+1}(\tau) - G_{j,k}(\tau)], \quad (2.20)$$

where the quantity $G_{j,k}$ is defined as

$$G_{j,k}(\tau) = \frac{1}{\Delta p} \left[ f(q_j, p_k, \tau) - f(q_{j-1}, p_k, \tau) \right] + \frac{p_k}{2} \left[ f(q_j, p_k, \tau) - f(q_{j-1}, p_k, \tau) \right] \quad (2.21)$$
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If we denote the known solution to the Vlasov equation on the grid and at time \( \tau_0 + \Delta \tau \) by \( f_{\text{vlas}}(q_j, p_k, \tau_0 + \Delta \tau) \), then we can find the solution to equation (2.19) by applying the formula

\[
f(q_j, p_k, \tau_0 + \Delta \tau) = f_{\text{vlas}}(q_j, p_k, \tau_0 + \Delta \tau) + \frac{D \Delta \tau}{\Delta p} \left[ G_{i,j+1}(\tau_0 + \Delta \tau) - G_{i,j}(\tau_0 + \Delta \tau) \right],
\]

where \( G \) in (2.21) uses \( f_{\text{vlas}} \). This completes the algorithm for finding the solution to the Vlasov-Fokker-Planck equation for the time step \( \tau_0 \rightarrow \tau_0 + \Delta \tau \).

2.2 Numerical Algorithm for Evaluating the Collective Force Due to CSR

In this section we describe an algorithm (appendix A of [4]) for evaluating the collective force due to CSR

\[
F_1(q, f(\cdot, \tau)) = -\omega_0 \sum_{n=-\infty}^{\infty} \exp \left( in \frac{\sigma_z q}{R} \right) Z(n, n\omega_0) \lambda_n \left( \frac{\tau}{\omega_s} \right).
\]

(2.23)

First, we consider the Fourier transform of the line density \( \lambda_n \left( \frac{\tau}{\omega_s} \right) \). Using equation (1.23) and the relationship \( \theta = \frac{z}{R} = \frac{(\sigma_z q)}{R} \), we can rewrite equation (1.20) as

\[
\lambda_n \left( \frac{\tau}{\omega_s} \right) = \frac{1}{2\pi} \int_{-\pi R/\sigma_z}^{\pi R/\sigma_z} e^{-inq \sigma_z / R} \rho(q) dq = \frac{1}{2\pi} \int_{-\kappa}^{\kappa} e^{-inq \sigma_z / R} \rho(q) dq.
\]

(2.24)

The charge density \( \rho(q, \tau) \) has compact support \([-\kappa, \kappa]\), with \( \kappa \) being much smaller than \( \pi R/\sigma_z \). A typical value for \( \kappa \) might be anywhere around six or seven for our own purposes. Since we have to compute the Fourier transform of the line density at every time step, we should employ the FFT to save computing time.
Recall that the FFT supplies the sum
\[
\frac{1}{J} \sum_{j=0}^{J-1} e^{-2\pi i m j/J} f(\theta_j), \quad m = 0, 1, \ldots, J - 1, \quad \theta_j = 2\pi j/J,
\] (2.25)
which is the result of using the trapezoidal rule to approximate the transform of a periodic function \( f(\theta) \),
\[
f_m = \frac{1}{2\pi} \int_0^{2\pi} e^{-im\theta} f(\theta) \, d\theta, \quad \Delta \theta = 2\pi/J.
\] (2.26)
Let’s first approximate the integral of an arbitrary continuous function \( h(q) \) over the interval \([-\kappa, \kappa]\). The trapezoidal rule gives us
\[
\int_{-\kappa}^{\kappa} h(q) \, dq = \sum_{j=0}^{2N-1} \int_{q_j}^{q_{j+1}} h(q) \, dq
\]
\[
= \sum_{j=0}^{2N-1} \left( \frac{h(q_{j+1}) + h(q_j)\Delta}{2} + \mathcal{O}(\Delta^3) \right)
\] (2.27)
\[
= \sum_{j=0}^{2N-1} h(q_j)\Delta + \frac{1}{2}[h(q_{2N}) - h(q_0)]\Delta + \mathcal{O}(\Delta^3)
\] (2.28)
\[
= \frac{\kappa}{N} \sum_{j=0}^{2N-1} h(q_j) + \mathcal{O}(\Delta),
\] (2.29)
where \( \Delta = \Delta q = \kappa/N \) is the grid step size. With \( h(q) = e^{-inq\sigma_z/R}\rho(q) \), we have
\[
\lambda_n \left( \frac{\tau}{\omega_s} \right) \approx \frac{\kappa}{2\pi N} \sum_{j=0}^{2N-1} e^{-inq_j\sigma_z/R}\rho(q_j)
\] (2.30)
\[
= \kappa e^{i\pi n/\Delta n} \frac{2N-1}{2\pi N} \sum_{j=0}^{2N-1} e^{-i\pi jn/(N\Delta n)}\rho(q_j), \quad \Delta n = \frac{\pi R}{\sigma_z\kappa}.
\] (2.31)
To convince the reader that the sum can be expressed in this form, we write the argument of the exponential as
\[
-\frac{inq_j\sigma_z}{R} = -in \left( \frac{j\kappa}{N} - \kappa \right) \frac{\sigma_z}{R} = -i \frac{\pi jn}{N\Delta n} + i \frac{\pi n}{\Delta n}.
\] (2.32)
To put this in the standard form for the FFT, we adjust \( \kappa \) (from its original value) to make \( \Delta n \) an integer. Since the ratio defining \( \Delta n \) is typically large compared to
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1, the minimum required adjustment of $\kappa$ is small. After adjusting $\kappa$, the sum takes the standard form for the FFT for values of $n$ which are integer multiples of $\Delta n$. With $n = m\Delta n$ we have

$$
\lambda_{m\Delta n} = \frac{\kappa e^{i\pi m}}{2\pi N} \sum_{j=0}^{2N-1} e^{-i2\pi mj/(2N)} \rho(q_j)
$$

(2.33)

$$
= \frac{\kappa(-1)^m}{\pi} \frac{1}{J} \sum_{j=0}^{J-1} (e^{-i2\pi j/J})^{m_j} \rho(q_j),
$$

(2.34)

$$
m = 0, 1, \cdots, N; \quad J = 2N.
$$

(2.35)

The upper limit on $m$ comes from the Nyquist rule, which says that a mode $m$ is meaningful only if the phase $2\pi mj/J$ does not change by more than $\pi$ when $j$ changes by one unit. This sum is now in the standard form for the FFT, so we can now use an efficient FFT code to evaluate it.

Now we focus our attention on evaluating the collective force in (1.18). First, we need to guess a maximum value for the number of modes $n$. Linear coasting beam theory states that modes near the maximum of the quantity $|Z_n/n|$ are likely to become unstable at high current. Therefore, it would make sense that $n$ should go well beyond the point where $|Z_n/n|$ is a maximum. It turns out that if we take $n = N\Delta n$, we easily surpass this point.

Now we consider an interpolative scheme for evaluating a general sum of the form

$$
S(\theta) = \sum_{m=m_0\Delta n}^{N\Delta n} e^{im\theta} g(m) = e^{iN\Delta n\theta} g(N\Delta n)
$$

(2.36)

$$
+ \sum_{m=m_0}^{N-1} \sum_{k=0}^{\Delta n-1} e^{i(m\Delta n+k)\theta} g(m\Delta n + k).
$$

(2.37)

We write $g(m\Delta n + k)$ as a low order polynomial in $k$ for $k \in [0, \Delta n - 1]$. This is the polynomial obtained from interpolating values $g(m'\Delta n)$ for $m'$ near $m$. The sum on $k$ can then be carried out in terms of sums of geometric series and their derivatives with respect to $\theta$. 

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We state the result for linear interpolation,

\[ g(m\Delta n + k) = g(m\Delta n) + \frac{g((m+1)\Delta n) - g(m\Delta n)}{\Delta n}k. \]  

(2.38)

After a fairly long calculation, one obtains

\[ S(\theta) = \frac{1}{\Delta n} \left( 1 - \cos \Delta n \theta \right) \sum_{m=m_0}^{N-1} g(m\Delta n)e^{im\Delta n \theta} + B(\theta), \]  

(2.39)

where the boundary term \( B(\theta) \) has the form

\[ B(\theta) = e^{i\Delta n \theta} g(N\Delta n) + \left[ \frac{1}{e^{\theta} - 1} + \frac{1 - e^{-i\Delta n \theta}}{2\Delta n(1 - \cos \theta)} \right] \]  

\times \left[ e^{i\Delta n \theta} g(N\Delta n) - e^{i m_0 \Delta n \theta} g(m_0 \Delta n) \right]. \]  

(2.40)

(2.41)

It turns out that for our application \( B(\theta) = 0 \).

We now compute (1.18) with the help of (2.39). We define \( g(n) = Z_n \lambda_n (\tau / \omega_s) \) and note that \( g(n) = g(-n)^* \). This says that the real part of \( g(n) \) is even in \( n \), and the imaginary part of \( g(n) \) is odd in \( n \). This allows us to eliminate the negative \( n \) in (1.18). We assume that \( g(N\Delta n) \) is negligible, and we know that \( Z(0) = 0 \). This gives us \( g(0) = g(N\Delta n) = 0 \). The collective force due to CSR can now be written as

\[ F_1(q_j, f(\cdot, \tau)) = -2\omega_0 \Re \sum_{n=0}^{N\Delta n} e^{inq_j \sigma_z / R} g(n) \]  

(2.42)

\[ = a(j) \Re \sum_{m=0}^{N} (-1)^m e^{imq_j / N} g(m\Delta n) \]  

(2.43)

\[ = a(j) \Re \sum_{m=0}^{2N-1} e^{i2\pi mj / (2N)} g_m, \]  

(2.44)

where

\[ a(j) = -\frac{2\omega_0}{\Delta n} \frac{1 - \cos(q_j \pi / \kappa)}{1 - \cos(q_j \sigma_z / R)}. \]  

(2.45)
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and

\[
g_m = \begin{cases} 
(-1)^m g(m\Delta n), & m = 0, \ldots, N \\
0, & m = N + 1, \ldots, 2N - 1.
\end{cases} \tag{2.46}
\]

By using this zero-padding technique, we get the result of an inverse FFT of length \(2N\) for all \(j = 0, 1, \ldots, 2N\).

This algorithm for finding the collective force due to CSR works well. However, we choose to use a modified algorithm due to B. Warnock, which calculates the Fourier transform of the line density in smaller steps of \(n\). The idea is to first choose the resolution for the desired step in \(n\), which we call \(\Delta n\). Typical values for \(\Delta n\) might be anywhere from 20 to 40. Next, we choose the parameter \(N\) to be an integer multiple of \(\widetilde{n}\), i.e. \(N = k\widetilde{n}\) for some positive integer \(k\). Recall that \(N\) is used to define the resolution of our two-dimensional grid. We also choose \(\Delta n = \text{int}[(\pi R)/(\sigma_z \kappa)]]\) to be an integer multiple of \(\widetilde{n}\), i.e. \(\Delta n = l\widetilde{n}\) for some positive integer \(l\). This will force us to choose an appropriate initial value for \(\kappa\) to give us the desired \(\Delta n\).

When finding \(\Delta n\) we take the integer part a ratio involving \(\kappa\). In turn, this will adjust \(\kappa\) from it’s original value to give \(\kappa = (\pi R)/(\sigma_z \Delta n)\). It is not necessary to choose \(\Delta n\) to be an integer multiple of \(\widetilde{n}\). However, if we want the length of the FFT to be a product of small primes (FFT algorithms work best when the length is a product of small primes), it is a good idea to play around with \(\kappa\) to make \(\Delta n\) an integer multiple of \(\widetilde{n}\). To illustrate this technique, let’s set \(\widetilde{n} = 40\). Then we can choose \(N = 200\) and \(\Delta n = 200\). If \(\Delta n = 200\), we should let \(\kappa = 6.46\) or some other value very close to 6.46. One can check that \(\text{int}[(\pi R)/(\sigma_z \kappa)] = 200\), using the values \(R = 2.037\), \(\sigma_z = 0.004946\), and \(\kappa = 6.46\). With \(\Delta n = 200\), we adjust \(\kappa\) from its original value to give us \(\kappa \approx 6.46929\).

The next step is to evaluate \(\lambda_n(\tau/\omega_s)\) at integer multiples of \(\widetilde{n}\), i.e. \(n = m\widetilde{n}\).
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Using equation (2.30) with \( n = m \Delta \tilde{n} \), we get

\[
\lambda_{m\Delta \tilde{n}} \left( \frac{\tau}{\omega_s} \right) = \frac{Ke^{i\pi m \Delta \tilde{n}/\Delta n}}{2\pi N} \sum_{j=0}^{2N-1} \exp\left[(-2\pi i)mj/(2kl \Delta \tilde{n})\right] \rho(q_j),
\]

(2.47)

where \( N = k \Delta \tilde{n} \) and \( \Delta n = l \Delta \tilde{n} \). Once again, the FFT supplies the sum

\[
\sum_{j=0}^{N-1} \exp((-2\pi i)mj/N) \rho(q_j)
\]

(2.48)

for a periodic function \( \rho(q) \). In equation (2.47), we have \( N = 2kl \Delta \tilde{n} \), but \( j \) run from 0 to \( 2N - 1 \), and not 0 to \( N - 1 \). Therefore, we use a zero-padding technique to perform an FFT of the correct length,

\[
\tilde{\rho}(q_j) = \begin{cases} 
\rho(q_j), & j = 0, \cdots, 2N - 1 \\
0, & j = 2N, \cdots, N - 1
\end{cases}
\]

(2.49)

We can then evaluate

\[
\lambda_{m\Delta \tilde{n}} \left( \frac{\tau}{\omega_s} \right) = \frac{Ke^{i\pi m \Delta \tilde{n}/\Delta n}}{2\pi N} \sum_{j=0}^{N-1} \exp\left[(-2\pi i)mj/N\right] \tilde{\rho}(q_j),
\]

(2.50)

where \( N = 2kl \Delta \tilde{n} \). If we use the values in the proceeding paragraph, we get \( N = 2000 \) for the length of this sum. For the range in \( m \), the Nyquist rule gives us \( m = 0, 1, \cdots, N/2 \). Finally, if we take \( n_{\text{max}} = N \Delta \tilde{n} \) for the maximum value of \( n \) in the collective force sum, we can proceed as before and evaluate the collective force using a linear interpolation approximation provided by equation (2.39), where \( m \) ranges from 0 to \( N \).
Chapter 3

Numerical Results: Part I

In this chapter we present some numerical results for the SLC Damping Ring. We take a uniform Cartesian grid in two-dimensional phase space extending from $-\kappa$ to $\kappa$ in both directions, where $\kappa = 6.46$. We choose $\Delta n = 40$ and $N = 200$ for a $401 \times 401$ grid and take 1024 time steps per synchrotron period. All of the results presented here were done for 600 periods.

We use the parameters given in table 1.1 and a wake potential function $W(q)$ provided by K. Bane which is discussed in [6]. To get the values of $W(q)$ on the grid, we use a simple linear interpolation of Bane’s wake potential. The units of measure for $W(q)$ is given in V/pC (volts per picoCoulomb). Thus, the corresponding units for $I$ is given by

$$\frac{I}{N} = \frac{\epsilon^2}{2\pi v_s \sigma_E} = 2.63 \times 10^{-12} \text{pC/V}. \quad (3.1)$$

Figure 3.1 is a graph of the wake potential on the grid for $q$, ranging from $-2\kappa$ to $2\kappa$. For the SLAC machine, the CSR has a much smaller effect than the machine wake. Therefore, the exact equilibrium for the combined case, which is given by (1.32) with the impedance converted to a wake potential and then added to the machine wake, is approximately the equilibrium for the case of machine wake alone. For either case,
a run at any current $I$ begins with this Haïssinski solution. A separate program is used to evaluate the Haïssinski equilibrium on the same two-dimensional grid, which uses Simpson’s rule to approximate the integrals.

### 3.1 Machine Wake Only

In this section we present results for a collective force $F(q, f(\cdot, \tau)) = F_2(q, f(\cdot, \tau))$ with $F_1(q, f(\cdot, \tau)) = 0$. Our first task is to find a threshold current $I$ which is an indicator for the onset of unstable solutions to the VFP equation. For currents below this threshold current, the solutions are stable. Whereas, for currents above this threshold current, instabilities start to arise after a significant amount of time. Figures 3.2, 3.3, and 3.4 show plots of the rms bunch length $\sigma_q$ and $\sigma_p$ versus time for the respective currents $I = 0.04, 0.045, 0.04975$ pC/V.

At $I = 0.04$, the Haïssinski equilibrium is apparently stable; the computed distrib-
Chapter 3. Numerical Results: Part I

Figure 3.2: rms bunch length $\sigma_q$ and rms energy spread $\sigma_p$ for $I = 0.04$

Figure 3.3: rms bunch length $\sigma_q$ and rms energy spread $\sigma_p$ for $I = 0.045$

ution $f(q, p, \tau)$ is indistinguishable to the eye from the original Haïssinski equilibrium for all times up to 600 periods. However, a closer inspection of the values for $\sigma_q$ shows that the final value is somewhat smaller than the original value. We believe that the calculation of the equilibrium solution is quite accurate, and that this decrease is due to the approximation errors in our Vlasov solver. A more accurate integration algorithm developed by M. Venturini is consistent with this interpretation.

For $I = 0.045$, small oscillations start to show up at the largest times. This indicates the onset of instability, and we call this value of $I$ the threshold current. We expect oscillations of higher amplitude as $I$ is increased even further. For values
of $I$ less than threshold, we expect the distribution to remain stable out to at least 600 periods. Of course approximation errors can propagate and eventually lead to unreliable results.

At currents just above the threshold, the time-domain algorithm gives sinusoidal oscillations of $\sigma_q$ and $\sigma_p$ at constant amplitude after a long transient period. For instance, at $I = 0.04975$ this behavior sets in after about 480 periods. The frequency of these oscillations is extremely large, a bit lower than $2\omega_s$. At higher currents, there is a slow transition to a mode in which these sinusoidal oscillations have a slowly varying amplitude. This particular mode is frequently referred to as the “bursting” or
Chapter 3. Numerical Results: Part I

"sawtooth" mode, and the instability associated with this mode is called a sawtooth instability. Figures 3.5, 3.6 and 3.7 exhibit sawtooth instabilities. By looking at the graphs of $\sigma_q$ and $\sigma_p$, one might conclude that $I = 0.055$ seems to be near the sawtooth mode threshold, i.e. for currents below 0.055 the sawtooth mode is no longer present. At this current, an asymptotic periodicity is suggested, but it is not yet clearly defined after 600 periods. It is clear that for $I = 0.07, 0.08$ a clear asymptotic periodicity starts to emerge after about 200 periods. For $I = 0.08$, the noisy behavior around 70 periods is actually quite smooth when observed on an expanded time scale.
Chapter 3. Numerical Results: Part I

At a sufficiently high current above threshold, the charge distribution $\rho(q)$ and the phase space density $f(q, p, \tau)$ start to develop some structure. However, even for $I = 0.08$, total charge is conserved to about one part in $10^5$, and the distribution is very small at the edges (around $10^{-7}$ or smaller). Figure 3.8 shows the final phase space distribution (top figure) compared with the initial Haïssinski distribution (bottom figure) for $I = 0.08$. The Haïssinski distribution is quite smooth, and looks like a leaning Gaussian. After 600 periods, the final phase space distribution has a complicated structure near the bottom of the distribution. Figure 3.9 shows the final charge density and initial Haïssinski for this same current. After 600 periods, the Haïssinski decreases in height and spreads out near the bottom.

Figures 3.10 and 3.11 show the charge distribution for $I = 0.07$ after 385, 405 and 495 synchrotron periods. These times correspond to when $\sigma_q$ (or $\sigma_p$) is either at a maximum or a minimum. The first two plots look very similar with the charge density increasing in height after 405 periods. After 495 periods, the charge density develops shoulders and increases in height slightly further. The density seems to be leaning slightly to the right in each one of the three plots. This is probably a consequence of the initial Haïssinski’s leaning structure.
Figure 3.8: final and initial phase space distributions for $I = 0.08$
Figure 3.9: final charge density (norm=0.9999794) with the Haïssinski for $I = 0.08$

Figure 3.10: charge density for $I = 0.08$ after 385 and 405 periods (norm=0.999986309 and 0.999985883)

Figure 3.11: charge density for $I = 0.08$ after 495 periods (norm=0.999983447)
3.2 Machine Wake and CSR

In this section we study the collective effects with both the machine wake and CSR; 
\[ F(q, f(\cdot, \tau)) = F_1(q, f(\cdot, \tau)) + F_2(q, f(\cdot, \tau)) \]. Like in the previous section, we first look for a threshold current \( I \) to determine the onset of instability. Figures 3.12, 3.13, and 3.14 compare the rms bunch length \( \sigma_q \) in the case of machine wake with the combined case for the currents \( I = 0.04, 0.045, 0.04975 \) pC/V. For each figure, the left subfigure corresponds to the case of wake only, and the right subfigure corresponds to the combined case.

For \( I = 0.04 \), one cannot distinguish between the two graphs. This leads us to conclude that we’re below the threshold current for the combined case. However for \( I = 0.045 \), the distribution on the right has larger fluctuations for long times than the one on the left. This leads us to conclude that CSR apparently has some small effect on the stability of solutions. This is consistent with the belief that the CSR has a smaller effect than the wake potential in the SLAC damping ring. For \( I = 0.045 \), we are clearly above the threshold current. Thus, the threshold should lie between 0.04 and 0.045, which is slightly lower than the threshold \( I = 0.045 \) for the case of machine wake alone.

At higher currents, we see a slow transition to a sawtooth instability. Figures 3.15 and 3.16 compare the two cases for \( I = 0.07, 0.08 \). For the combined case, there is a stronger instability in the numerical solution of the VFP equation, i.e. there is a greater range in \( \sigma_q \). Also, the onset of sawtooth instability sets in earlier. The noisy behavior around 80 periods is actually quite smooth when expanded on a small scale for time.

At our highest current \( I = 0.08 \), the charge distribution \( \rho(q) \) and the phase space density \( f(q, p, \tau) \) have some complicated structure as in the case of machine wake alone. Figure 3.17 compares the final phase space distributions for both cases.
Chapter 3. Numerical Results: Part I

The one on top is from the case of wake only and the one on bottom is from the combined case. For the combined case, the lower half of the distribution starts to spread out, while the top becomes skinnier. At higher currents, the collective force will kick the distribution out of our grid after a short time. If we wish to study the numerical solutions for higher currents, we will need to use a larger grid. However, it is likely that we could also run into other problems for such high currents. Figure 3.18 compares the final charge densities for $I = 0.08$. The charge density for the combined case is taller and starts to develop shoulders which come and go with large times. Although the two charge densities look very different, they might just be out of phase. In other words, at some time the charge density for the case of wake alone might also start to develop shoulders.

In conclusion, the CSR has an effect on the numerical solution of the VFP equation. However, the instability threshold for the combined case is only slightly lower than the case of machine wake acting alone.
Chapter 3. Numerical Results: Part I

Figure 3.12: comparison of rms bunch length $\sigma_q$ for $I = 0.04$

Figure 3.13: comparison of rms bunch length $\sigma_q$ for $I = 0.045$

Figure 3.14: comparison of rms bunch length $\sigma_q$ for $I = 0.04975$
Figure 3.15: comparison of rms bunch length $\sigma_q$ for $I = 0.07$

Figure 3.16: comparison of rms bunch length $\sigma_q$ for $I = 0.08$
Figure 3.17: comparison of the final phase space distributions for $I = 0.08$
Figure 3.18: comparison of the final charge densities for $I = 0.08$ (norm=0.9999794 and 0.99998007)
Chapter 4

Numerical Results: Part II

Now we focus our attention on another machine known as the VUV Storage Ring. We solve the same Vlasov-Fokker-Planck equation (1.17), but with a different set of parameters. The VUV Storage Ring is a light source located at Brookhaven National Laboratory. Following the information given by [8], we first discuss what a light source is.

A light source is an accelerator, which provides a source for intense beams of x-rays and ultraviolet radiation. The x-rays and ultraviolet radiation make possible both basic and applied research in a variety of scientific disciplines. The primary interest in a light source is the wavelengths in the x-ray and ultraviolet ranges, although, infrared waves are also included. The light source makes use of the fact that if electrons are constrained to move on a circular path, they will produce synchrotron light (or synchrotron radiation) due to their inward acceleration. Today’s synchrotron-based light source is a large, roughly circular machine, which accelerates particle bunches close to the speed of light. The components of the accelerator consist of an electron gun, one or more injector accelerators, and a storage ring. The injector accelerator increases the particle energy from a zero energy up to some final
value. The purpose of the storage ring is to store the electrons for a few hours while experiments are conducted. This contrasts with the damping ring, where particle bunches only circulate for a few milliseconds before they are injected into the main accelerator. To counter the loss of charge in the storage ring, the ring is refilled several times a day.

The VUV storage ring was one of the first second-generation light sources to operate in the world. The ring is filled five times per day to a fill current of 850 mA. Its purpose is to generate intense ultraviolet “light” for scientific and technological research. As the path of the particle bunch bends, synchrotron light is emitted tangentially to the curved path and streams down beamlines to the instruments used to conduct experiments.

In this chapter, we follow the same setup as in Chapter 1, and we consider the particle dynamics due to the effects of CSR and a machine wake potential. We use the same numerical algorithm outlined in Chapter 2. The only difference from Chapter 3 is the values for the parameters. The table on the next page lists the values all relevant parameters which we use.

Once again, we take a uniform Cartesian grid in two-dimensional phase space extending from $-\kappa$ to $\kappa$ in both directions, where $\kappa = 7.05$. We choose $\Delta n = 4$ and $N$ for a $401 \times 401$ grid and take 1024 time steps per synchrotron period. All of our runs were done out to 250 periods.

Since we are using a different machine, we can no longer use K. Bane’s wake potential. We hope to approximate the machine wake in the VUV Storage Ring by a broad-band resonator model. In this model we deal with the impedance formalism of the machine wake rather than the wake potential formalism. As stated earlier, we can easily go from one form to the other by taking a Fourier transform. Following
Chapter 4. Numerical Results: Part II

<table>
<thead>
<tr>
<th>Quantity</th>
<th>Symbol</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>Nominal Energy</td>
<td>$E_0$</td>
<td>$7.37 \times 10^8 \text{ eV} = 0.737 \text{ GeV}$</td>
</tr>
<tr>
<td>Synchrotron Frequency</td>
<td>$\omega_s = 2\pi f_s$</td>
<td>$75 \times 10^3 \text{ Hz} = 75 \text{ KHz}$</td>
</tr>
<tr>
<td>Revolution Frequency</td>
<td>$f_0$</td>
<td>$5.9 \times 10^6 \text{ Hz} = 5.9 \text{ MHz}$</td>
</tr>
<tr>
<td>Bend Radius</td>
<td>$R$</td>
<td>$1.91 \text{ m}$</td>
</tr>
<tr>
<td>Momentum Compaction</td>
<td>$\alpha$</td>
<td>$0.0245$</td>
</tr>
<tr>
<td>Synchrotron Tune</td>
<td>$\nu_s = f_s/f_0$</td>
<td>$0.002$</td>
</tr>
<tr>
<td>Lorentz Factor</td>
<td>$\gamma_0 = E_0/(mc^2)$</td>
<td>$1442$</td>
</tr>
<tr>
<td>Slip Factor</td>
<td>$\beta_0 = \sqrt{1 - 1/\gamma_0^2}$</td>
<td>$1$</td>
</tr>
<tr>
<td>Energy Damping Time</td>
<td>$\tau_d$</td>
<td>$10 \text{ ms} = 10 \times 10^{-3} \text{ s}$</td>
</tr>
<tr>
<td>Diffusion Constant</td>
<td>$D = 2\beta = 2/(\omega_r \tau_d)$</td>
<td>$0.00265$</td>
</tr>
<tr>
<td>Bunch Length</td>
<td>$\sigma_z$</td>
<td>$0.05 \text{ m}$</td>
</tr>
<tr>
<td>Energy Spread</td>
<td>$\sigma_E$</td>
<td>$3.69 \times 10^5 \text{ eV} = 0.369 \text{ MeV}$</td>
</tr>
</tbody>
</table>

Table 4.1: fixed parameters for the VUV Storage Ring

Chao and others, the broad-band impedance for the wake is approximated by

$$Z(\omega) = \frac{R}{1 + iQ(\omega - \omega_r)}$$

(4.1)

where $Q$ is a dimensionless parameter known as the quality factor, $R$ is the resistance typically given in Ohms, and $\omega_r = 2\pi f_r$ is the resonator frequency. We approximate (4.1) using the Breit-Wigner form in nuclear physics to obtain

$$Z(\omega) = \frac{i}{2} \frac{\Gamma}{R} \left[ \frac{1}{\omega - \omega_r - i\Gamma/2} + \frac{1}{\omega + \omega_r - i\Gamma/2} \right],$$

(4.2)

where $\Gamma/2 = \omega_r/(2Q)$ is measured in Hertz. Since the code that we use is setup for a wake formalism for the machine wake, we need to take the Fourier transform of (4.2), using residue theory, to evaluate the integral, to obtain the corresponding wake potential. However, (4.2) will prove to be useful later on when we evaluate the radiated power due to the machine wake. The wake potential for our machine wake is given by

$$W(q) = \begin{cases} 0, & q > 0 \\ \exp(-\frac{\Gamma}{2} |\tau|) \cos(\omega_r \tau), & q < 0 \end{cases},$$

(4.3)
where $\tau = q\sigma_z/(\beta c)$. When $q = 0$ we take half the value of $W$ when $q = 0^-$. The factor $10^{-12}$ arises, because we wish to express $W$ in units of V/pC. Following B. Podobedov’s comments, measurements have been done to obtain the following values: $R/n = 1.8 \Omega$, $Q = 1.118$, $f_r = c/\lambda_r = (3 \cdot 10^8 \text{m/s})/(84 \text{mm}) = 3.57 \text{GHz}$, $\omega_r = 2\pi f_r = 22.4 \text{GHz}$, $n = f_r/f_0 = (3.57 \cdot 10^9 \text{s}^{-1})/[(3 \cdot 10^8 \text{m/s})/(51 \text{m})] = 607$ for the ring circumference of 51 m.

Figure 4.1 shows the VUV wake potential using (4.3) and the values given in the previous paragraph, with $\Gamma = 20.0 \text{GHz}$ and $R = 1092 \Omega$. Comparing this with figure 3.1, we see that the maximum value for the SLAC machine wake is around 20 times larger than the maximum value for the VUV machine wake. This suggests that the effect from the machine wake may be much smaller in the VUV ring than in the SLAC ring. However, like the SLAC damping ring, the effect from the machine wake still dominates the effect from the CSR.
Chapter 4. Numerical Results: Part II

4.1 CSR Only

In this section we consider the effects of CSR alone. As before, we set out to find the threshold current $I$ for the onset of instability. For all runs in this section, we start with a Gaussian distribution. We still use the idea that the equilibrium solution to the VFP equation with the effects of CSR only is approximately equal to the equilibrium for the unperturbed dynamics, which is a Gaussian. Figures (4.2), (4.3) show plots of $\sigma_q$ versus time for the currents $I = 6.25, 6.3, 12.53$ respectively. A plot of $\sigma_p$ is also shown for $I = 12.53$.

For $I = 6.25$ we see that $\sigma_q$ is relatively constant out to 250 periods. Thus, we are below threshold for this current. However, for $I = 6.3$ we start to see some small microbursts initially, then the bursts start to slowly decay with time. We might conclude that we are now above threshold, and we should see stronger instabilities for higher currents. For $I = 12.53$, we are well above threshold, and we start to see a very strong instability for small times. There are periodic bursts in $\sigma_q$ that slowly decay with time. It seems that $\sigma_q$ is starting to approach some constant value as time increases. If we integrate for a longer time, we suspect that we would see a horizontal line suggesting that the solution to the VFP equation is approaching

![Figure 4.2: rms bunch length $\sigma_q$ for $I = 6.25, 6.3$](image-url)
a new equilibrium. The plot of $\sigma_p$ is nearly identical to the plot of $\sigma_q$ so that this effect is not related to a rotation in phase space. Rather, it is probably related to the initial distribution being symmetrical.

The results obtained so far are quite different than the results obtained for the SLAC runs. For these runs, we initially see large fluctuations in the plots of $\sigma_q$ and $\sigma_p$ with a gradual decay to smaller fluctuations. In the SLAC damping ring, we saw the opposite effect; small fluctuations initially with a gradual growth to large fluctuations.

Another interesting quantity to look at is the radiated power coming from the particle bunch. As electrons circulate, they radiate power. Following the discussion in [7], we define the coherent radiated power as

$$P^{coh}_n(t) = 2(eN\omega_0)^2 \sum_n \text{Re}Z_n|\lambda_n(t)|^2$$  \hspace{1cm} (4.4)

However, we look at the ratio of coherent to incoherent radiated power and not coherent radiated power alone. The incoherent power is defined as

$$P^{incoh}_n(t) = \frac{2N(e\omega_0)^2}{(2\pi)^2} \sum_n \text{Re}Z_n$$  \hspace{1cm} (4.5)

In these two equations $Z_n$ is the CSR impedance and should not be confused with
Chapter 4. Numerical Results: Part II

\( Z(\omega) \), which is the VUV wake impedance.

Recall that from chapter 2 we only found \( \lambda_n \) for integer multiples of \( \tilde{\Delta}n \), i.e. for \( n = m \tilde{\Delta}n \). Thus, we use a linear interpolation to evaluate (4.4) and (4.5). In other words, we take the ratio of the two sums

\[
P_{coh}^{m\Delta\tilde{n}}(t) = 2(eN\omega_0)^2 \tilde{\Delta}n \sum_m \text{Re}Z_{m\Delta\tilde{n}}|\lambda_{m\Delta\tilde{n}}(t)|^2, \tag{4.6}
\]

\[
P_{incoh}^{m\Delta\tilde{n}}(t) = \frac{2N(e\omega_0)^2}{(2\pi)^2} \tilde{\Delta}n \sum_m \text{Re}Z_{m\Delta\tilde{n}}. \tag{4.7}
\]

Following the discussion in [7], the physical description for the radiated power can be described as follows. The vacuum chamber gives an exponential suppression of radiation at wavelengths \( \lambda \) greater than a “shielding cutoff” \( \lambda_0 \approx 2h(h/R)^{1/2} \), where \( h \) is the chamber height and \( R \) is the bending radius. Initially, there are small microstructures in the bunch which give small Fourier components with wavelength below this shielding cutoff. Above the current threshold, these Fourier components build up exponentially because the impedance from synchrotron radiation is very large below the cutoff value. This leads to a burst of radiation, which is limited in duration by a quick smoothing of the phase space distribution. Continued exponential growth is restricted by the intrinsic nonlinearity of self-consistent many-particle dynamics, which also contributes to the smoothing of the phase space distribution through the quick generation of a relatively large spectrum of Fourier modes. Within a short number of periods, the microstructures have almost disappeared, the overall bunch length increases, and the burst of radiation is finished. Damping and diffusion from the incoherent radiation gradually reduce the bunch length and energy spread, restoring the conditions for instability and another burst of radiation.

Figures 4.4, 4.5 show the ratios for the currents \( I = 6.3, 12.53 \). In both figures we have periodic large bursts in power with small microstructures in between these bursts. It is interesting to note that the times when we are at the peak values of these
bursts correspond to the times when we are at the minimum values of the sawtooth bursts in $\sigma_q$. It is evident that as we increase the current parameter $I$, the frequency and amplitude of the bursts increase as well.

We conclude this section by looking at the charge density $\rho(q,t)$ for times of the first burst occurring between 26 and 29 synchrotron periods. There are 11 frames of $\rho(q)$ during this time period. The 11 frames are ordered from left to right and from top to bottom. The last plot on the second page is an enlargement of the radiated power in figure 4.5, which should serve as a guide for the times when $\rho(q)$ is plotted.

After 27.0625 periods, very small microstructures start to appear to the right of the maximum point on the graph of $\rho(q)$. They go away after 27.34375 periods, but then reappear after 27.625 periods. At this particular time, larger microstructures are clearly present. On the next frame, they are still quite large, but go away thereafter. These microstructures are also known as a microbunching in the charge distribution. We can also see that the support of the charge density changes over time.
Chapter 4. Numerical Results: Part II

Figure 4.4: ratio of coherent to incoherent radiated power for $I = 6.3$

Figure 4.5: ratio of coherent to incoherent radiated power for $I = 12.53$
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Figure 4.6: charge density after 26.21875 and 26.5 periods

Figure 4.7: charge density after 26.78125 and 27.0625 periods

Figure 4.8: charge density after 27.34375 and 27.625 periods
Chapter 4. Numerical Results: Part II

Figure 4.9: charge density after 27.90625 and 28.1875 periods

Figure 4.10: charge density after 28.46875 and 28.5625 periods

Figure 4.11: charge density after 28.75 periods and first radiated burst for $I = 12.53$
4.2 Machine Wake Only

The effects from machine wake are much stronger than the effects from the CSR. This was also the case for the SLAC damping rings. However, as we saw earlier, the machine wake in the VUV ring is considerably smaller than the machine wake in the SLAC damping rings. In this section, we expect to see the threshold current to be much smaller than the threshold current in the previous section.

Figures 4.12 and 4.13 show the rms bunch length for the currents $I = 2.35, 2.40, 2.45, 2.50$. By referring to these figures, we define $I = 2.40$ to be the current threshold, which is significantly lower than the current threshold from the previous section.

In contrast to the SLAC machine, we do not see sawtooth instabilities at currents significantly above this threshold. Figure 4.14 shows the rms bunch length for $I = 4.5, 5.0$. From looking at the figure, we see that the instability in $\sigma_q$ has no periodicity whatsoever. Unfortunately, we do not have the ability to generate results for currents above $I = 5.0$ because the Newton iteration method, which is used to generate the Haïssinski equilibrium, does not converge for higher currents.

The final charge densities for $I = 4.5, 5.0$ are shown in Figure 4.15. One can see that the Haïssinski density develops some shoulders and moves from left to right as the current parameter $I$ increases. The density also become wider near the top as $I$ increases from 4.0 to 5.0.

We already have a formula for the coherent power coming from the CSR. The coherent power coming from the machine wake is given by

$$-f_0(eN)^2 \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \rho(q, \tau)\rho(q', \tau)W(q - q') dq dq'. \quad (4.8)$$

It is interesting to note that the coherent radiated power coming from the machine wake is very small. Figure 4.16 shows the coherent power for $I = 5.0$. One can see
that it is essentially zero. As we will see in the next section, the coherent radiated power coming from the combined CSR and machine wake is much larger than the power shown here.
Chapter 4. Numerical Results: Part II

Figure 4.12: rms bunch length $\sigma_q$ for $I = 2.35, 2.40$

Figure 4.13: rms bunch length $\sigma_q$ for $I = 2.45, 2.50$

Figure 4.14: rms bunch length $\sigma_q$ for $I = 4.5, 5.0$
Figure 4.15: final charge density for $I = 4.5, 5.0$ (norm=0.99998548 and 0.99970204)

Figure 4.16: coherent power for $I = 5.0$
4.3 Machine Wake and CSR

It turns out that the combined effect of the machine wake and the CSR does not affect the threshold very much. Figures 4.17 and 4.18 show $\sigma_q$ for various currents.

One may be curious to know why we see stronger oscillations for $I = 2.40$ than for $I = 2.45$. We do not understand this phenomenon, but it may be an artifact of numerical error associated with the integration method we use. Nevertheless, we define the threshold to be around $I = 2.40$.

At above threshold, we see some instabilities qualitatively similar to the instabilities found in the previous section. In figures 4.19 and 4.20, $\sigma_q$ is shown for $I = 3.5, 4.0, 4.5, 5.0$. Comparing figure 4.20 to the previous section, we can see that the CSR does have some effect on the solution. With the effect of CSR, large oscillations in $\sigma_q$ last for a longer period of time. Also, for $I = 5.0$ the onset of strong instabilities sets in sooner than in the case of machine wake alone.

The coherent power for $I = 0.05$ is shown in figure 4.21. One may notice that there is a very strong burst after 1 or 2 periods. The precise location of this first strong burst is around 1.08 synchrotron periods. This strong burst may indicate some interesting behavior in the charge density. Plotting the charge density after 1.08 periods reveals a wild structure consisting of three “peaks”. To convince ourselves that this structure may possibly be real and not just some consequence of numerical error in our integration scheme, we decide to increase the resolution of the grid from $401^2$ grid points to $801^2$ grid points. We also decrease the time step by a factor of six to give us 6144 time steps per period. Figures 4.22, 4.23 and 4.24 show the results of three numerical runs done out to 1.08 synchrotron periods. When the machine wake acts alone, we see several peaks in the charge density, but no microstructure. The charge density appears to be quite smooth, and the charge normalization is conserved to one part in $10^{-5}$. When we add the effect of the CSR, we see some
microstructure develop in two areas of the charge density plot. With the appearance of the microstructure, the charge density is still conserved quite well. It is the conjecture of B. Warnock that the machine wake creates some high frequency modes which are then amplified by the CSR. This might be what causes the microstructure to appear in the charge density. We also plotted the final charge density with the CSR alone just to convince ourselves that there is not a fundamental error with the integration code.

In conclusion, the CSR does not have much affect on the instability threshold, but it does have a visible affect for currents well above threshold. The CSR causes the particle bunch to radiate power, which significantly alters the structure of the charge density and gives rise to microstructures. Although the CSR is much smaller than the machine wake, the radiated power due to the CSR is much larger than the radiated power due to the machine wake.

The next step in this project is to compare the results obtained from a more accurate integrator. M. Venturini has developed a code which uses a cubic Hermite interpolation method (Yabe’s method) to integrate the VFP equation more accurately. One possible objective is to duplicate his results found in [7] using this integration scheme. Another possible objective is to further investigate the microstructures that start to show up in the charge density whenever a large burst of radiation is emitted.
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Figure 4.17: rms bunch length $\sigma_q$ for $I = 2.35, 2.4$

Figure 4.18: rms bunch length $\sigma_q$ for $I = 2.45, 2.5$

Figure 4.19: rms bunch length $\sigma_q$ for $I = 3.5, 4.0$
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Figure 4.20: rms bunch length $\sigma_q$ for $I = 4.5, 5.0$

Figure 4.21: coherent power for $I = 5.0$
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Figure 4.22: final charge density after 1.08 periods, $I = 5.0$, CSR and machine wake combined (norm=1.00006202)

Figure 4.23: final charge density after 1.08 periods, $I = 5.0$, machine wake alone (norm=0.999999804)
Figure 4.24: final charge density after 1.08 periods, $I = 5.0$, CSR alone (norm=1)
References


References