Stat Inf HW4 solutions 5/13/08

7.63 Let $a = \tau^2/(\tau^2+1)$, so the Bayes estimator is $\delta^\pi(x) = ax$. Then $R(\mu, \delta^\pi) = (a-1)^2\mu^2 + a^2$. As τ^2 increases, $R(\mu, \delta^\pi)$ becomes flatter.

8.6 a.

$$\begin{aligned}\lambda(\mathbf{x},\mathbf{y}) &= \frac{\sup_{\Theta_0} L(\theta|\mathbf{x},\mathbf{y})}{\sup_{\Theta} L(\theta|\mathbf{x},\mathbf{y})} = \frac{\sup_\theta \prod_{i=1}^n \frac{1}{\theta}e^{-x_i/\theta} \prod_{j=1}^m \frac{1}{\theta}e^{-y_j/\theta}}{\sup_{\theta,\mu} \prod_{i=1}^n \frac{1}{\theta}e^{-x_i/\theta} \prod_{j=1}^m \frac{1}{\mu}e^{-y_j/\mu}} \\ &= \frac{\sup_\theta \frac{1}{\theta^{m+n}} \exp\left\{-\left(\sum_{i=1}^n x_i + \sum_{j=1}^m y_j\right)\big/\theta\right\}}{\sup_{\theta,\mu} \frac{1}{\theta^n}\exp\left\{-\sum_{i=1}^n x_i/\theta\right\} \frac{1}{\mu^m}\exp\left\{-\sum_{j=1}^m y_j/\mu\right\}}.\end{aligned}$$

Differentiation will show that in the numerator $\hat\theta_0 = (\sum_i x_i + \sum_j y_j)/(n+m)$, while in the denominator $\hat\theta = \bar x$ and $\hat\mu = \bar y$. Therefore,

$$\begin{aligned}\lambda(\mathbf{x},\mathbf{y}) &= \frac{\left(\frac{n+m}{\sum_i x_i + \sum_j y_j}\right)^{n+m} \exp\left\{-\left(\frac{n+m}{\sum_i x_i + \sum_j y_j}\right)\left(\sum_i x_i + \sum_j y_j\right)\right\}}{\left(\frac{n}{\sum_i x_i}\right)^n \exp\left\{-\left(\frac{n}{\sum_i x_i}\right)\sum_i x_i\right\}\left(\frac{m}{\sum_j y_j}\right)^m \exp\left\{-\left(\frac{m}{\sum_j y_j}\right)\sum_j y_j\right\}} \\ &= \frac{(n+m)^{n+m}}{n^n m^m} \frac{\left(\sum_i x_i\right)^n \left(\sum_j y_j\right)^m}{\left(\sum_i x_i + \sum_j y_j\right)^{n+m}}.\end{aligned}$$

And the LRT is to reject H_0 if $\lambda(\mathbf{x},\mathbf{y}) \le c$.

b.

$$\lambda = \frac{(n+m)^{n+m}}{n^n m^m}\left(\frac{\sum_i x_i}{\sum_i x_i + \sum_j y_j}\right)^n \left(\frac{\sum_j y_j}{\sum_i x_i + \sum_j y_j}\right)^m = \frac{(n+m)^{n+m}}{n^n m^m} T^n (1-T)^m.$$

Therefore λ is a function of T. λ is a unimodal function of T which is maximized when $T = \frac{n}{m+n}$. Rejection for $\lambda \le c$ is equivalent to rejection for $T \le a$ or $T \ge b$, where a and b are constants that satisfy $a^n(1-a)^m = b^n(1-b)^m$.

c. When H_0 is true, $\sum_i X_i \sim \text{gamma}(n,\theta)$ and $\sum_j Y_j \sim \text{gamma}(m,\theta)$ and they are independent. So by an extension of Exercise 4.19b, $T \sim \text{beta}(n,m)$.

8.10 Let $Y = \sum_i X_i$. The posterior distribution of $\lambda|y$ is gamma $(y+\alpha, \beta/(\beta+1))$.

a.

$$P(\lambda \le \lambda_0|y) = \frac{(\beta+1)^{y+\alpha}}{\Gamma(y+\alpha)\beta^{y+\alpha}} \int_0^{\lambda_0} t^{y+\alpha-1} e^{-t(\beta+1)/\beta}\, dt.$$

$P(\lambda > \lambda_0|y) = 1 - P(\lambda \le \lambda_0|y)$.

b. Because $\beta/(\beta+1)$ is a scale parameter in the posterior distribution, $(2(\beta+1)\lambda/\beta)|y$ has a gamma$(y+\alpha, 2)$ distribution. If 2α is an integer, this is a $\chi^2_{2y+2\alpha}$ distribution. So, for $\alpha = 5/2$ and $\beta = 2$,

$$P(\lambda \le \lambda_0|y) = P\left(\frac{2(\beta+1)\lambda}{\beta} \le \frac{2(\beta+1)\lambda_0}{\beta}\,\middle|\, y\right) = P(\chi^2_{2y+5} \le 3\lambda_0).$$

8.12 a. For $H_0\colon \mu \le 0$ vs. $H_1 : \mu > 0$ the LRT is to reject H$_0$ if $\bar{x} > c\sigma/\sqrt{n}$ (Example 8.3.3). For $\alpha = .05$ take $c = 1.645$. The power function is

$$\beta(\mu) = P\left(\frac{\bar{X}-\mu}{\sigma/\sqrt{n}} > 1.645 - \frac{\mu}{\sigma/\sqrt{n}}\right) = P\left(Z > 1.645 - \frac{\sqrt{n}\mu}{\sigma}\right).$$

Note that the power will equal .5 when $\mu = 1.645\sigma/\sqrt{n}$.

b. For $H_0\colon \mu = 0$ vs. $H_A\colon \mu \ne 0$ the LRT is to reject H_0 if $|\bar{x}| > c\sigma/\sqrt{n}$ (Example 8.2.2). For $\alpha = .05$ take $c = 1.96$. The power function is

$$\beta(\mu) = P\left(-1.96 - \sqrt{n}\mu/\sigma \le Z \le 1.96 + \sqrt{n}\mu/\sigma\right).$$

In this case, $\mu = \pm 1.96\sigma/\sqrt{n}$ gives power of approximately .5.

8.14 The CLT tells us that $Z = (\sum_i X_i - np)/\sqrt{np(1-p)}$ is approximately n$(0,1)$. For a test that rejects H_0 when $\sum_i X_i > c$, we need to find c and n to satisfy

$$P\left(Z > \frac{c - n(.49)}{\sqrt{n(.49)(.51)}}\right) = .01 \quad \text{and} \quad P\left(Z > \frac{c - n(.51)}{\sqrt{n(.51)(.49)}}\right) = .99.$$

We thus want

$$\frac{c - n(.49)}{\sqrt{n(.49)(.51)}} = 2.33 \quad \text{and} \quad \frac{c - n(.51)}{\sqrt{n(.51)(.49)}} = -2.33.$$

Solving these equations gives $n = 13{,}567$ and $c = 6{,}783.5$.

8.16 a.

$$\begin{aligned} \text{Size} &= P(\text{reject } H_0 \mid H_0 \text{ is true}) = 1 \Rightarrow \text{Type I error} = 1. \\ \text{Power} &= P(\text{reject } H_0 \mid H_A \text{ is true}) = 1 \Rightarrow \text{Type II error} = 0. \end{aligned}$$

b.

$$\begin{aligned} \text{Size} &= P(\text{reject } H_0 \mid H_0 \text{ is true}) = 0 \Rightarrow \text{Type I error} = 0. \\ \text{Power} &= P(\text{reject } H_0 \mid H_A \text{ is true}) = 0 \Rightarrow \text{Type II error} = 1. \end{aligned}$$

8.18 a.

$$\begin{aligned} \beta(\theta) &= P_\theta\left(\frac{|\bar{X}-\theta_0|}{\sigma/\sqrt{n}} > c\right) = 1 - P_\theta\left(\frac{|\bar{X}-\theta_0|}{\sigma/\sqrt{n}} \le c\right) \\ &= 1 - P_\theta\left(-\frac{c\sigma}{\sqrt{n}} \le \bar{X}-\theta_0 \le \frac{c\sigma}{\sqrt{n}}\right) \\ &= 1 - P_\theta\left(\frac{-c\sigma/\sqrt{n} + \theta_0 - \theta}{\sigma/\sqrt{n}} \le \frac{\bar{X}-\theta}{\sigma/\sqrt{n}} \le \frac{c\sigma/\sqrt{n} + \theta_0 - \theta}{\sigma/\sqrt{n}}\right) \\ &= 1 - P\left(-c + \frac{\theta_0-\theta}{\sigma/\sqrt{n}} \le Z \le c + \frac{\theta_0-\theta}{\sigma/\sqrt{n}}\right) \\ &= 1 + \Phi\left(-c + \frac{\theta_0-\theta}{\sigma/\sqrt{n}}\right) - \Phi\left(c + \frac{\theta_0-\theta}{\sigma/\sqrt{n}}\right), \end{aligned}$$

where $Z \sim$ n$(0,1)$ and Φ is the standard normal cdf.

b. The size is $.05 = \beta(\theta_0) = 1 + \Phi(-c) - \Phi(c)$ which implies $c = 1.96$. The power (1 − type II error) is

$$.75 \le \beta(\theta_0+\sigma) = 1 + \Phi(-c-\sqrt{n}) - \Phi(c-\sqrt{n}) = 1 + \underbrace{\Phi(-1.96-\sqrt{n})}_{\approx 0} - \Phi(1.96-\sqrt{n}).$$

$\Phi(-.675) \approx .25$ implies $1.96 - \sqrt{n} = -.675$ implies $n = 6.943 \approx 7$.

8.20 By the Neyman-Pearson Lemma, the UMP test rejects for large values of $f(x|H_1)/f(x|H_0)$. Computing this ratio we obtain

x	1	2	3	4	5	6	7
$\frac{f(x\|H_1)}{f(x\|H_0)}$	6	5	4	3	2	1	.84

The ratio is decreasing in x. So rejecting for large values of $f(x|H_1)/f(x|H_0)$ corresponds to rejecting for small values of x. To get a size α test, we need to choose c so that $P(X \le c|H_0) = \alpha$. The value $c = 4$ gives the UMP size $\alpha = .04$ test. The Type II error probability is $P(X = 5, 6, 7|H_1) = .82$.

8.21 The proof is the same with integrals replaced by sums.

8.22 a. From Corollary 8.3.13 we can base the test on $\sum_i X_i$, the sufficient statistic. Let $Y = \sum_i X_i \sim \text{binomial}(10, p)$ and let $f(y|p)$ denote the pmf of Y. By Corollary 8.3.13, a test that rejects if $f(y|1/4)/f(y|1/2) > k$ is UMP of its size. By Exercise 8.25c, the ratio $f(y|1/2)/f(y|1/4)$ is increasing in y. So the ratio $f(y|1/4)/f(y|1/2)$ is decreasing in y, and rejecting for large value of the ratio is equivalent to rejecting for small values of y. To get $\alpha = .0547$, we must find c such that $P(Y \le c|p = 1/2) = .0547$. Trying values $c = 0, 1, \ldots$, we find that for $c = 2$, $P(Y \le 2|p = 1/2) = .0547$. So the test that rejects if $Y \le 2$ is the UMP size $\alpha = .0547$ test. The power of the test is $P(Y \le 2|p = 1/4) \approx .526$.

b. The size of the test is $P(Y \ge 6|p = 1/2) = \sum_{k=6}^{10} \binom{10}{k} \left(\frac{1}{2}\right)^k \left(\frac{1}{2}\right)^{10-k} \approx .377$. The power function is $\beta(\theta) = \sum_{k=6}^{10} \binom{10}{k} \theta^k (1-\theta)^{10-k}$

c. There is a nonrandomized UMP test for all α levels corresponding to the probabilities $P(Y \le i|p = 1/2)$, where i is an integer. For $n = 10$, α can have any of the values 0, $\frac{1}{1024}$, $\frac{11}{1024}$, $\frac{56}{1024}$, $\frac{176}{1024}$, $\frac{386}{1024}$, $\frac{638}{1024}$, $\frac{848}{1024}$, $\frac{968}{1024}$, $\frac{1013}{1024}$, $\frac{1023}{1024}$, and 1.

8.30 a. For $\theta_2 > \theta_1 > 0$, the likelihood ratio and its derivative are

$$\frac{f(x|\theta_2)}{f(x|\theta_1)} = \frac{\theta_2}{\theta_1}\frac{\theta_1^2+x^2}{\theta_2^2+x^2} \quad \text{and} \quad \frac{d}{dx}\frac{f(x|\theta_2)}{f(x|\theta_1)} = \frac{\theta_2}{\theta_1}\frac{\theta_2^2-\theta_1^2}{(\theta_2^2+x^2)^2}x.$$

The sign of the derivative is the same as the sign of x (recall, $\theta_2^2 - \theta_1^2 > 0$), which changes sign. Hence the ratio is not monotone.

b. Because $f(x|\theta) = (\theta/\pi)(\theta^2 + |x|^2)^{-1}$, $Y = |X|$ is sufficient. Its pdf is

$$f(y|\theta) = \frac{2\theta}{\pi}\frac{1}{\theta^2+y^2}, \quad y > 0.$$

Differentiating as above, the sign of the derivative is the same as the sign of y, which is positive. Hence the family has MLR.

9.10

9.13 a. For $Y = -(\log X)^{-1}$, the pdf of Y is $f_Y(y) = \frac{\theta}{y^2}e^{-\theta/y}$, $0 < y < \infty$, and

$$P(Y/2 \le \theta \le Y) = \int_\theta^{2\theta} \frac{\theta}{y^2} e^{-\theta/y} dy = e^{-\theta/y}\Big|_\theta^{2\theta} = e^{-1/2} - e^{-1} = .239.$$

b. Since $f_X(x) = \theta x^{\theta-1}$, $0 < x < 1$, $T = X^\theta$ is a good guess at a pivot, and it is since $f_T(t) = 1$, $0 < t < 1$. Thus a pivotal interval is formed from $P(a < X^\theta < b) = b - a$ and is

$$\left\{\theta\colon \frac{\log b}{\log x} \le \theta \le \frac{\log a}{\log x}\right\}.$$

Since $X^\theta \sim$ uniform(0, 1), the interval will have confidence .239 as long as $b - a = .239$.

c. The interval in part a) is a special case of the one in part b). To find the best interval, we minimize $\log b - \log a$ subject to $b - a = 1 - \alpha$, or $b = 1 - \alpha + a$. Thus we want to minimize $\log(1 - \alpha + a) - \log a = \log\left(1 + \frac{1-\alpha}{a}\right)$, which is minimized by taking a as big as possible. Thus, take $b = 1$ and $a = \alpha$, and the best $1 - \alpha$ pivotal interval is $\left\{\theta\colon 0 \le \theta \le \frac{\log \alpha}{\log x}\right\}$. Thus the interval in part a) is nonoptimal. A shorter interval with confidence coefficient .239 is $\{\theta\colon 0 \le \theta \le \log(1 - .239)/\log(x)\}$.

9.17 a. Since $X - \theta \sim$ uniform$(-1/2, 1/2)$, $P(a \le X - \theta \le b) = b - a$. Any a and b satisfying $b = a + 1 - \alpha$ will do. One choice is $a = -\frac{1}{2} + \frac{\alpha}{2}$, $b = \frac{1}{2} - \frac{\alpha}{2}$.

b. Since $T = X/\theta$ has pdf $f(t) = 2t$, $0 \le t \le 1$,

$$P(a \le X/\theta \le b) = \int_a^b 2t\, dt = b^2 - a^2.$$

Any a and b satisfying $b^2 = a^2 + 1 - \alpha$ will do. One choice is $a = \sqrt{\alpha/2}$, $b = \sqrt{1 - \alpha/2}$.

9.23 a. The LRT statistic for $H_0\colon \lambda = \lambda_0$ versus $H_1\colon \lambda \ne \lambda_0$ is

$$g(y) = e^{-n\lambda_0}(n\lambda_0)^y / e^{-n\hat\lambda}(n\hat\lambda)^y,$$

where $Y = \sum X_i \sim$ Poisson$(n\lambda)$ and $\hat\lambda = y/n$. The acceptance region for this test is $A(\lambda_0) = \{y\colon g(y) > c(\lambda_0))$ where $c(\lambda_0)$ is chosen so that $P(Y \in A(\lambda_0)) \ge 1 - \alpha$. $g(y)$ is a unimodal function of y so $A(\lambda_0)$ is an interval of y values. Consider constructing $A(\lambda_0)$ for each $\lambda_0 > 0$. Then, for a fixed y, there will be a smallest λ_0, call it $a(y)$, and a largest λ_0, call it $b(y)$, such that $y \in A(\lambda_0)$. The confidence interval for λ is then $C(y) = (a(y), b(y))$. The values $a(y)$ and $b(y)$ are not expressible in closed form. They can be determined by a numerical search, constructing $A(\lambda_0)$ for different values of λ_0 and determining those values for which $y \in A(\lambda_0)$. (Jay Beder of the University of Wisconsin, Milwaukee, reminds us that since c is a function of λ, the resulting confidence set need not be a highest density region of a likelihood function. This is an example of the effect of the imposition of one type of inference (frequentist) on another theory (likelihood).)

b. The procedure in part a) was carried out for $y = 558$ and the confidence interval was found to be (57.78, 66.45). For the confidence interval in Example 9.2.15, we need the values $\chi^2_{1116,.95} = 1039.444$ and $\chi^2_{1118,.05} = 1196.899$. This confidence interval is $(1039.444/18, 1196.899/18) = (57.75, 66.49)$. The two confidence intervals are virtually the same.