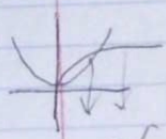
Last time delta method
Today: review, 5 6 maybe 6.1

Another example for delta method.

Let $X \sim \exp$ with mean β.

What is $E[\sqrt{X}]$ and $Var(\sqrt{X})$.

First-order delta method approximation gives $E[\sqrt{X}] \approx \sqrt{E(X)} = \sqrt{\beta}$



underestimates $E[\sqrt{X}]$.

$$E[\sqrt{X}] = \int_0^\infty g(x) f_X(x)\, dx$$

$$= \frac{1}{\beta} \int_0^\infty \sqrt{x}\, e^{-x/\beta}\, dx$$

$$= \frac{1}{\beta} \int_0^\infty \underbrace{x^{1/2} e^{-x/\beta}}_{\text{looks like gamma}(\frac{3}{2}, \beta)}\, dx$$

Recall for a gamma, density

$$\frac{1}{\Gamma(\alpha)\beta^\alpha} x^{\alpha-1} e^{-x/\beta}$$

$$\frac{1}{\beta} \int_0^\infty \frac{\Gamma(\alpha)\beta^{3/2}}{\Gamma(\alpha)\beta^{3/2}} x^{3/2-1} e^{-x/\beta}\, dx$$

$$> \text{gamma}(1.5) = \frac{\Gamma(\frac{3}{2})\beta^{3/2}}{\beta^1} = \sqrt{\beta} \cdot \Gamma\left(\tfrac{3}{2}\right) \approx .886\sqrt{\beta}$$

$$Var(\sqrt{x}) \approx [g'(\mu)]^2 \underbrace{Var(x)}_{\beta^2}$$

$g(y) = \sqrt{y}$

$g'(y) \frac{1}{2\sqrt{y}}$

$$g'(y)\Big|_{y=\mu} = \frac{1}{2\sqrt{\mu}} = \frac{1}{2\sqrt{\beta}}$$

So $var(\sqrt{x}) \approx [g'(\mu)]^2 \, Var(x)$

$$= \left(\frac{1}{2\sqrt{\beta}}\right)^2 \beta^2$$

$$= \frac{1}{4\beta} \cdot \beta^2 = \frac{\beta}{4}$$

Could check using

$$Var(\sqrt{x}) = E[(\sqrt{x})^2] - (E[\sqrt{x}])^2$$

$$= E(\cancel{x}\, x) - (E(\sqrt{x}))^2$$

$$= \beta - \left(\Gamma\left(\tfrac{3}{2}\right)\sqrt{\beta}\right)^2$$

$$= \beta - \left(\Gamma\left(\tfrac{3}{2}\right)\right)^2 \beta$$

$$= \beta\left(1 - \left(\Gamma(3/2)\right)^2\right)$$

Generally, how to compute or estimate

$$E[g(X)].$$

Strategies

(1) $E[g(X)] = \int g(x) f_X(x)\, dx$

(2) delta method: $E[g(X)] \approx g(\mu)$ (f

or second-order $\approx g(\mu) + \frac{g''(\mu)}{2} \text{var}$

$$g(X) = g(\mu) + g'(\mu)(X-\mu) + \frac{g''(\mu)}{2}(X$$

$$E[g(X)] = g(\mu) + \frac{g''(\mu)}{2} \text{Var}(X) + R_2$$

(3) let $Y = g(X)$.

Derive the density for Y

Use $E[Y] = \int f_Y(y) \cdot y\, dy$

(4) Use simulation.

In R

```
> x <- rexp(10000)
> mean(sqrt(x))
```

Generally, how to compute or estimate

$$E[g(X)].$$

Strategies

(1) $E[g(X)] = \int g(x) f_X(x)\, dx$

(2) delta method: $E[g(X)] \approx g(\mu)$ (first order)

or second-order $\approx g(\mu) + \frac{g''(\mu)}{2} var(X)$

$$g(X) = g(\mu) + g'(\mu)(X-\mu) + \frac{g''(\mu)}{2}(X-\mu)^2 + R_2$$

$$E[g(X)] = g(\mu) + \frac{g''(\mu)}{2} Var(X)$$

(3) let $Y = g(X)$.

Derive the density for Y

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(4)

Simulating from a nonstand density

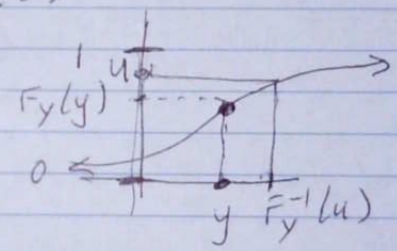
Example. ~~Suppose~~

Suppose Y has density

$$f_Y(y) = \frac{1}{y^2} I\left(\frac{1}{2} \le y \le 1\right)$$

To simulate Y which has density $f_Y(y)$ generate $u = U(0,1)$

Then let $Y = F_Y^{-1}(U)$



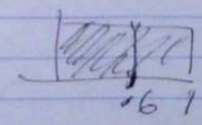
Also use cdf to check why this works.

$P[F_Y^{-1}(U) \le u]$

$= P[F_Y(F_Y^{-1}(U)) \le F_Y(u)]$

$= P[U \le F_Y(u)]$ For a $U(0,1)$ $P(U \le u) = u$

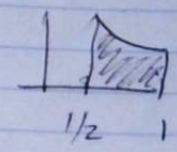
$= F_Y(u)$



let $X = U(1,2)$
$Y = 1/X$.

Then

$$f_Y(y) = \frac{1}{y^2} I(\tfrac{1}{2} \le y \le 1)$$

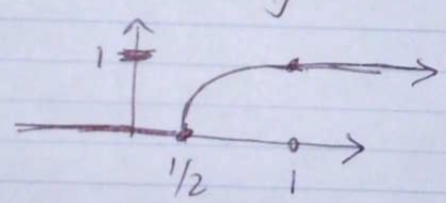


$$F_Y(y) = \int_{1/2}^{y} \frac{1}{t^2}\, dt$$

$$= -t^{-1}\Big|_{1/2}^{y}$$

$$= -\frac{1}{y} - -\frac{1}{1/2}$$

$$= 2 - \frac{1}{y} \qquad I(\tfrac{1}{2} \le y \le 1)$$



For the inverse cdf

$$x = 2 - \frac{1}{y} \quad \text{solve for } y$$

$$\frac{1}{y} = 2 - x \Rightarrow y = \frac{1}{2-x}$$

$$F_Y^{-1}(y) = \frac{1}{2-y} \qquad Y = F_Y^{-1}(U) = \frac{1}{2-U}$$

to generate Y use
```
> u <- runif(0,1)
> y <- 1/(2-u)
```

(6)

Suppose $X \sim exp$. What is density of $Y = \frac{1}{1+X}$

```
> x<-rexp(1000)
> hist(x/(1+x))
```

$F_X^{-1}(U)$

Ch. 6 Principles of data reduction.

If $X_1, \dots, X_n$ is a random sample, then $T(\underline{X})$, a statistic of $\underline{X}$, is often a summary of the data.

Examples: 5-number summary
(min, 25%, 50%, 75%, max)

box plot

$\bar{X}$

median

mode

$X_{(n)} / X_{(1)}$

Usually, we're interested in statistics that preserve the useful information in the data for making inferences about parameters.

Example. Suppose $X_1, X_2 \overset{iid}{\sim} \text{Pois}(\lambda)$.

We want to figure out/estimate λ.
If I just tell $\bar{X}$, is that enough?

Suppose $\bar{X} = 2$. Data could have been
$(0,4)$, $(2,2)$, $(1,3)$, $(3,1)$, $(4,0)$

A statistic partitions the set of all possible data sets.

$A_t = \{ \underline{x} : T(\underline{x}) = t \}$

For Poisson example, $T(\underline{X}) = \frac{X_1 + X_2}{2}$

$A_2 = \{(0,4), (1,3), (2,2), (3,1), (4,0)\}$

$A_1 = \{(0,2), (1,1), (2,0)\}$

$A_0 = \{(0,0)\}$

$A_{1/2} = \{(0,1), (1,0)\}$

Suppose $T(\underline{X}) = X_1$.

<u>Sufficiency Principle</u>. If $T(\underline{X})$ is a sufficient statistic (ss) (sufficient for θ), then any inference ~~of~~ about θ should depend on $\underline{X}$ only through $T(\underline{X})$

If $\underline{x}$ and $\underline{y}$ are two data sets where $T(\underline{x}) = T(\underline{y})$, then any inference about θ should be the same for $\underline{x}$ and $\underline{y}$.