Actual Final, 2016

Formula Sheet

$$Var(X) = E(X^2) - (E(X))^2, Cov(X, Y) = E[XY] - E[X]E[Y]$$

$$Cov\left(\sum_i X_i, \sum_j Y_j\right) = \sum_i \sum_j Cov(X_i, Y_j)$$

$$E\{E[X|Y]\} = E[X]$$

$$Var(X) = E\{Var(X|Y)\} + Var(E[X|Y])$$

Bayes Theorem:

$$P(A_i|B) = \frac{P(B|A_i)P(A_i)}{\sum_i P(B|A_i)P(A_i)}$$

Note that the order of problems does not reflect the order in which the material was covered in class and also is not necessarily arranged least to most difficult, so you might want to read the final first before deciding what problems to do first.

1. Let $S = \{1, 2, \dots, n\}$ be a sample space. How many subsets of the sample space have more than one outcome?

Ans. $2^n - n - 1$.

2. Let X have density

$$f_X(x) = 6x(1-x)\ I(0 < x < 1)$$

and let Y be a uniform(0,1) random variable, with X and Y being independent. Recall that for a uniform(0,1), $E[Y] = 1/2$ and $Var(Y) = 1/12$.

(a) Find the density for $Z = X + Y$. You can use any method you like.

(b) Find $Cov(Y, Z)$.

For convolutions, we can use either

$$f_Z(z) = \int_{-\infty}^{\infty} f_Y(y) f_X(z-y)\ dy$$

or

$$f_Z(z) = \int_{-\infty}^{\infty} f_X(x) f_Y(z-x)\ dx$$

The second version is more convenient becuase X has a more complicated density. The integrand is positive when

$$0 < x < 1 \quad \text{and } 0 < z - x < 1$$

Together, these imply that

$$x > \max(0, z-1)$$

and

$$x < \min(1, z)$$

or

$$f_Z(z) = \int_{\max(0,z-1)}^{\min(1,z)} 6x(1-x)\ dx$$

We also know that $0 < z < 2$. For $0 < z < 1$, we have $0 < x < z$, and for $1 < z < 2$, we have $z - 1 < x < 1$. Thus,

<u>For $0 < z < 1$</u>:

$$\begin{aligned} f_Z(z) &= \int_0^z 6x - 6x^2\ dx \\ &= 3x^2 - 2x^3 \Big|_0^z \\ &= 3z^2 - 2z^3 \end{aligned}$$

<u>For $0 < z < 1$</u>:

$$
\begin{aligned}
f_Z(z) &= \int_0^{1+z} 6x - 6x^2 \, dx \\
&= 3x^2 - 2x^3 \Big|_{z-1}^{1} \\
&= (3-2) - (3(z-1)^2 - 2(z-1)^3) \\
&= 1 - 3(z-1)^2 + 2(z-1)^3
\end{aligned}
$$

Finally, the density is

$$
f_Z(z) = \begin{cases} 3z^2 - 2z^3 & 0 < z < 1 \\ 1 - 3(z-1)^2 + 2(z-1)^3 & 1 \leq z < 2 \\ 0 & \text{otherwise} \end{cases}
$$

3. Your friend always uses either airline A or airline B. Airline A is twice as likely to be late as airline B. If your friend takes airline A 25% of the time and arrives late, what is the probability your friend took airline A?

Let A be the event that airline A is taken, and let B be the event that airline B is taken. Let L be the event that the plane is late.

Let $P(A) = 2p$ and $P(B) = p$. Then using Bayes' Theorem,

$$P(A|L) = \frac{P(L|A)P(A)}{P(L|A)P(A) + P(L|B)P(B)} = \frac{2p(1/4)}{2p(1/4) + p(3/4)} = \frac{2}{2+3} = \frac{2}{5}$$

4. Suppose $Y|X \sim \exp(X)$ and $X \sim \exp(\lambda)$ (i.e., $E[X] = 1/\lambda$ and $Var(X) = 1/\lambda^2$). Find

(a) $E[Y]$

Sol.

$$E[Y] = E\{E[Y|X]\} = E[1/X]$$

This expected value is actually infinite. Using the integral directly is difficult. One way to see this is to use the fact that for $x < 1$, $e^{-\lambda x} \geq e^{-\lambda}$.

$$\begin{aligned} E[X] &= \int_0^\infty \frac{\lambda}{x} e^{-\lambda x}\, dx \\ &\geq \int_0^1 \frac{\lambda}{x} e^{-\lambda x}\, dx \\ &\geq \lambda e^{-\lambda} \int_0^1 \frac{1}{x}\, dx \end{aligned}$$

The integral diverges and is infinite.

(b) $Var(Y)$

$$Var(Y) = E[Var(Y|X)] + Var(E[Y|X]) = E(1/X^2) + (E[1/X])^2$$

The variance is also infinite.

(c) For graduate students: Is Y an exponential random variable? Justify your answer.

No, because the expectation is infinite.

5. For undergraduates, do either (a) or (b) (your choice). For graduate students, do both.

(a) Let U_1 and U_2 be independent uniform(0,1) random variables. Find the joint density for $U = U_1 - U_2$ and $V = U_1$.

(b) Find the density for U. You can either integrate the joint density obtained in (a) or use the cdf method or do a modification of the convolution formula (you'd have to figure out how to modify the convolution formula appropriate for a subtraction of random variables).

Solutions.

Using the transformation $u = u_1 - u_2$ and $v = u_1$, the Jacobian is

$$\begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix} = \begin{vmatrix} \frac{1}{-1} & -\frac{x}{(x+y)^2} \\ 1 & 0 \end{vmatrix} = 1$$

The joint density is

$$\begin{aligned} f_{U,V}(u,v) &= f_{U_1,U_2}(u_1,u_2)|J|^{-1} \\ &= f_{U_1,U_2}(v, v-u) \\ &= f_{U_1}(v) f_{U_2}(v-u) I(0 < v < 1, 0 < v-u < 1) \\ &= 1 \cdot I(0 < v < 1, 0 < v-u < 1) \end{aligned}$$

To get the marginal density for U, we integrate the joing density with respect to v. The inequalities imply

$$v > \max(0,u), \quad v < \min(1, 1+u)$$

<u>For $-1 < u < 0$</u>:

$$\begin{aligned} f_U(u) &= \int_0^{1+u} dv \\ &= v\Big|_0^{1+u} \\ &= 1 + u \end{aligned}$$

<u>For $0 < u < 1$</u>:

$$\begin{aligned} f_U(u) &= \int_u^1 dv \\ &= v\Big|_u^1 \\ &= 1 - u \end{aligned}$$

The density is

$$f_U(u) = \begin{cases} 1+u & -1 < u < 0 \\ 1-u & 0 \le u < 1 \\ 0 & \text{otherwise} \end{cases}$$

6. Let X and Y have joint mass function

	Y		
X	1	2	3
1	1/9	1/9	1/9
2	2/9	2/9	2/9

(a) Are X and Y independent? Justify your answer.

Yes. We can check $P(X = i, Y = j) = P(X = i)P(Y = j)$ for all choices. For $i = 1$, $j = 1, 2, 3$, we have $P(X = 1, Y = j) = 1/9 = (1/3)(1/3) = P(X = 1)P(Y = j)$ for $j = 1, 2, 3$. For $i = 2$, $j = 1, 2, 3$, we have $P(X = 2, Y = j) = 2/9 = (2/3)(1/3) = P(X = 2)P(Y = j)$ for $j = 1, 2, 3$.

(b) What is $E[X]$?

$$E[X] = 1 \cdot P(X = 1) + 2 \cdot P(X = 2) = 1(1/3) + 2(2/3) = 5/3$$

(c) What is $Cov(X, Y)$?

Because X and Y are independent, the covariance is 0. No need to check further. Otherwise, use

$$Cov(X, Y) = E[XY] - E[X]E[Y]$$

where

$$E[XY] = \sum_{i=1}^{2} \sum_{j=1}^{3} ij P(X = i, Y = j)$$

7. (a) Six friends go to a restuarant and sit at a long rectangular table with three people on each side. Three of the friends wear glasses, and the other three wear contact lenses. If each person sits randomly, what is the probability that the friends wearing glasses all sit on the same side of the table?

There are different ways to approach it. One is that there are 6 positions, so 6! seating arrangements. However, there are $3! = 6$ ways for the people with glasses to sit together on the left, and 3! ways for the contact-wearers to sit together on the right. Because left and right don't matter, the desired probability is

$$2 \times \frac{3!3!}{6!} = 2\Big/\binom{6}{3} = 2/20 = 1/10$$

(b) For graduate students: answer part (a) when there are $2n$ friends, and n wear glasses and n wear contact lenses, and the friends with glasses all end up on the same side of the table.

$$2\Big/\binom{2n}{n}$$

8. For graduate students only (extra credit for undergrads). Name at least three relationships between distributions. An example (which you can't use towards the three) is that for $n \to \infty$ and $np \to \lambda$, a binomial distribution with parameters n and p approaches a Poisson distribution with rate λ. Depict the relationships in a chart like the one in the back of Casella and Berger.