Practice Test 2

Formula Sheet

$$Var(X) = E(X^2) - (E(X))^2, Cov(X,Y) = E[XY] - E[X]E[Y]$$

1. Let X have pdf

$$f_X(x) = \begin{cases} x/2, & 0 < x < 2 \\ 0, & \text{otherwise} \end{cases}$$

(a) Find $E[X]$

$$E[X] = \int_0^2 x^2/2 \, dx = x^3/6 \Big|_0^2 = 4/3$$

(b) Let $Y = \sqrt{X}$. Find $f_Y(y)$. Be sure to define $f_Y(y)$ for $-\infty < y < \infty$.

For $0 < y < \sqrt{2}$,

$$\begin{aligned} F_y(y) &= P(Y \le y) \\ &= P(\sqrt{X} \le y) \\ &= P(X \le y^2) \\ &= F_X(y^2) \\ \Rightarrow f_Y(y) &= \frac{d}{dy} F_X(y^2) \\ &= f_X(y^2) \cdot \frac{d}{dy} y^2 \\ &= y^2/2 \cdot 2y \\ &= y^3 \end{aligned}$$

Thus,

$$f_Y(y) = \begin{cases} y^3 & 0 < y < \sqrt{2} \\ 0 & \text{otherwise} \end{cases}$$

(c) Find Cov(X,Y). Hint: Use the fact that $Y = \sqrt{X}$ instead of trying to find the joint density for X and Y.

$$Cov(X,Y) = Cov(X, \sqrt{X}) = E[X \cdot \sqrt{X}] - E[X]E[\sqrt{X}]$$
$$E[X\sqrt{X}] = E[X^{3/2}] = \int_0^2 x^{5/2}/2 \, dx = (2/7)x^{7/2}/2\Big|_0^2 = (1/7)2^{7/2} = (8/7) \cdot \sqrt{2}$$
$$E[\sqrt{X}] = \int_0^2 x^{3/2}/2 \, dx = (2/5)x^{5/2}/2\Big|_0^2 = (1/5)2^{5/2} = (4/5) \cdot \sqrt{2}$$
$$\Rightarrow Cov(X,Y) = (8/7) \cdot \sqrt{2} - (4/3)(4/5) \cdot \sqrt{2} = \sqrt{2} \cdot ((8/7) - 16/15) = 0.1077496$$

Note that as X increases, $\sqrt{X}$ also increases, so it makes sense that the covariance is positive.

2. Let X and Y have joint density

$$f_{X,Y}(x,y) = \begin{cases} k(2x+y), & x, y > 0 \\ 0, & \text{otherwise} \end{cases}$$

Solution. I needed to put an upper limit on x and y for this problem to make sense. I'll use $0 < x, y < 1$ but you could use whatever finite limits you want to have a slight variation on the problem.

(a) Find k

$$k \int_0^1 \int_0^1 2x + y \, dxdy = 1$$
$$\Rightarrow k \int_0^1 x^2 + yx\Big|_0^1 dy = 1$$
$$\Rightarrow k \int_0^1 1 + y \, dy = 1$$
$$\Rightarrow k(y + y^2/2)\Big|_0^1 = 1$$
$$\Rightarrow k(1 + 1/2) = 1$$
$$\Rightarrow k = 2/3$$

(b) Find $f_X(x)$

for $0 < x < 1$,

$$\begin{aligned} f_X(x) &= \int_0^1 f_{X,Y}(x,y)dy \\ &= (2/3)\int_0^1 2x + y\ dy \\ &= (2/3)(2xy + y^2/2)\Big|_0^1 \\ &= (2/3)(2x + 1/2) \end{aligned}$$

Thus,

$$f_X(x) = (2/3)(2x + 1/2)I(0 < x < 1)$$

(c) Find $f_Y(y)$

for $0 < y < 1$,

$$\begin{aligned} f_Y(x) &= \int_0^1 f_{X,Y}(x,y)\ dx \\ &= (2/3)\int_0^1 2x + y\ dx \\ &= (2/3)(x^2 + xy)\Big|_0^1 \\ &= (2/3)(1 + y) \end{aligned}$$

Thus,

$$f_Y(y) = (2/3)(1 + y)I(0 < y < 1)$$

(d) Find Cov(X,Y)

$$
\begin{aligned}
E[XY] &= (2/3)\int_0^1\int_0^1 xy(2x+y)\;dxdy \\
&= (2/3)\int_0^1\int_0^1 2x^2y+xy^2\;dxdy \\
&= (2/3)\int_0^1 2x^3y/3+x^2y^2/2\Big|_0^1\;dy \\
&= (2/3)\int_0^1 2y/3+y^2/2\;dy \\
&= (2/3)\left(2y^2/6+y^3/6\Big|_0^1\right) \\
&= (2/3)(2/6+1/6) = 1/3 \\
E[X] &= (2/3)\int_0^1 2x^2+x/2\;dx \\
&= (2/3)(2x^2/3+x^2/4)\Big|_0^1 \\
&= (2/3)(2/3+1/4) = (2/3)(11/12) = 11/18 \\
E[Y] &= (2/3)\int_0^1 y+y^2\;dx \\
&= (2/3)(y^2/2+y^3/3)\Big|_0^1 \\
&= (2/3)(1/2+1/3) = 5/9 \\
Cov(X,Y) &= 1/3-(11/18)(5/9) = 54/162-55/162 = -1/162
\end{aligned}
$$

This suggests that as either X or Y go up, the other tends to go down by a tiny amount. This is not obvious from the mathematical description. However, they do not appear to be independent because the joint density does not look like a product of a function of x and a function of y.

(e) Find the density of the sum, $f_{X+Y}(a)$.

3. Let X and Y have joint mass function

	Y		
X	1	2	3
1	0.1	0.05	0.2
2	0.05	0.1	0.1
3	0.2	0.1	0.1

(a) Are X and Y independent? Justify your answer.

Solution. $P(X=1)=0.35$. $P(Y=1)=0.2$. $P(X=1,Y=1)=0.1 \neq P(X=1)P(Y=1)=0.07$. Thus, X and Y are not independent.

(b) Find the marginal distribution of X by specifying the probability mass function for X.

$$P(X=i)=\begin{cases} 0.35 & i=1 \\ 0.25 & i=2 \\ 0.4 & i=3 \\ 0 & \text{otherwise} \end{cases}$$

The values are obtained for $P(X=1)$ (as an example) by calculating

$$P(X=1)=P(X=1,Y=1)+P(X=1,Y=2)+P(X=1,Y=3)$$