Test 2

Formula Sheet

$$Var(X) = E(X^2) - (E(X))^2, Cov(X,Y) = E[XY] - E[X]E[Y]$$

1. Let X have pdf

$$f_X(x) = \begin{cases} 1/x^2, & x > 1 \\ 0, & \text{otherwise} \end{cases}$$

(a) Find the median of the distribution. This is the value m such that $\int_{-\infty}^{m} f(x)\,dx = 1/2$.

(b) Find $P(X > 5 | X > 4)$.

(c) Let $Y = 1/X$. Find $f_Y(y)$. Be sure to define $f_Y(y)$ for $-\infty < y < \infty$.

(d) Find $Cov(X,Y)$.

Solutions.

(a) First we'll find the CDF. The median is the value m such that $F(m) = 1/2$. The CDF is (for $x > 1$):

$$F_X(x) = \int_1^x t^{-2}\, dt - t^{-1}\Big|_1^x = 1 - 1/x$$

The median satisfies

$$1 - 1/x = 0.5 \Rightarrow 1/x = 0.5 \Rightarrow x = 1/0.5 \Rightarrow x = 2$$

So the median is $m = 2$.

(b) Here $P(X > x) = 1 - F_X(x) = 1 - (1 - 1/x) = 1/x$.

$$P(X > 5|X > 4) = \frac{P(X > 5, X > 4)}{P(X > 4)} = \frac{P(X > 5)}{P(X > 4)} = \frac{1/5}{1/4} = \frac{4}{5}$$

(c) Not that since $x > 1$, $0 < 1/x < 1$.

$$\begin{aligned}
F_Y(y) &= P(Y \leq y) \\
&= P(1/X \leq y) \\
&= P(X \geq 1/y) \\
&= 1 - F_X(1/y) \\
&\Rightarrow f_Y(y) = \frac{d}{dy} 1 - F_X(1/y) \\
&\Rightarrow f_Y(y) = f_Y(1/y)(1/y^2) \\
&\Rightarrow f_Y(y) = y^2/y^2 \\
&\Rightarrow f_Y(y) = 1
\end{aligned}$$

$$f_Y(y) = 1 \cdot (0 < y < 1)$$

so $Y = 1/X$ has a uniform distribution.

(d)

$$Cov(X, Y) = Cov(X, 1/X) = E[X/X] - E[X]E[Y] = E[1] - (\infty)(1/2) = -\infty$$

2. Let X and Y be independent where X is exponential with rate 1 and Y is exponential with rate 2:

$$f_X(x) = e^{-x}\ I(x > 0), \quad f_Y(y) = 2e^{-2y}\ I(y > 0)$$

Find the density for $Z = X + Y$.

Solution.

Using the convolution formula we have

$$f_Z(z) = \int_0^\infty f_Y(y) f_X(z - y)\ dy$$

where $y > 0$ and $z - y > 0$. The second inequality gives $y < z$, so the convolutions is (for $z > 0$)

$$\begin{aligned} f_Z(z) &= \int_0^\infty f_Y(y) f_X(z - y)\ dy \\ &= \int_0^z 2e^{-2y} e^{-(z-y)}\ dy \\ &= \int_0^z 2e^{-y} e^{-z}\ dy \\ &= 2e^{-z} \int_0^z e^{-y}\ dy \\ &= 2e^{-z}\left(1 - e^{-z}\right) \\ &= 2e^{-z} - 2e^{-2z} \end{aligned}$$

Finally

$$f_Z(z) = (2e^{-z} - 2e^{-2z}) I(z > 0)$$

3. Let X and Y have joint mass function

		Y	
X	1	2	3
1	1/9	1/18	1/18
2	1/18	1/9	2/9
3	1/18	2/9	1/9

(a) Are X and Y independent? Justify your answer.

(b) What is $E[X]$?

Solutions.

(a) They are not independent. For independence, $P(X = i, Y = j) = P(X = i)P(Y = j)$ for all choices of i and j. It is sufficient to show that one choice fails. For example,

$$P(X = 2, Y = 3) = \frac{2}{9} \neq \frac{7}{18} \times \frac{7}{18} = P(X = 2)P(Y = 3)$$

For $E[X]$, we need the marginal distribution for X, which is

$$P(X = i) = \begin{cases} \frac{4}{18} & i = 1 \\ \frac{7}{18} & i = 2 \\ \frac{7}{18} & i = 3 \\ 0 & \text{otherwise} \end{cases}$$

$$E[X] = (1)(4/18) + (2)(7/18) + (3)(7/18) = \frac{4 + 14 + 21}{18} = \frac{39}{18} = \frac{13}{6}$$