HW1, for MATH441, STAT461, STAT561, due September 1st

1. How many ways are there to rearrange the letters in $ALBUQUERQUE$?

Solution. In the solution, the denominator is based on 1 A, 1 B, 2 Es, 1 L, 1 R, 2Qs, and 3 Us.

$$\frac{11!}{1!1!2!1!1!2!3!}$$

2. (a) How many license plates are possible that have two digits (0–9) followed by UNM followed by another two digits? Assume that digits are allowed to repeat.

Solution.

$$10^2 \times 10^2 = 10^4$$

(b) How many license plates are possible that have three digits followed by three letters (with 26 letters in the alphabet)?

Solution.

$$10^3 \times 26^3$$

(c) How many license plates are possible that have three digits and three letters in any order?

Solution. First we choose the 3 locations (out of 6 possible) for the numbers. Letters will be in the remaining positions.

$$\binom{6}{3} 10^3 \times 26^3$$

3. Consider passwords with 10 characters where each character is either an upper case or lower case letter (numbers and special characters are not allowed).

(a) How many such passwords are there that have five lower case and five upper case letters?
(b) How many such passwords are there that have at least one lower case and at least one upper case letter?
Solution. (a) This is very similar to problem 2c.

$$\binom{10}{5} 26^5 \times 26^5$$

(b) There is more than one way to do this, but the easiest is to throw away the bad passwords:

$$52^{10} - 26^{10} - 26^{10} \approx 1.442728 \times 10^{17}$$

4. A department has 10 full, 8 associate, and 7 assistant professors.

(a) How many ways can a committee with two of each type of professor be formed?

(b) Suppose one of the full professors is married to one of the associate professors. How many ways can a committee be formed with two of each type of professor but the two people in the married couple cannot both serve on the committee?

Solution. (a)

$$\binom{10}{2}\binom{8}{2}\binom{7}{2}$$

(b) Either enumerate all ways of having such committees (neither individual from the married couple, the married full professor but not associate, or the married associate but not full), or throw away bad committees.

$$\binom{9}{2}\binom{7}{2}\binom{7}{2}+\binom{9}{1}\binom{7}{2}\binom{7}{2}+\binom{9}{2}\binom{7}{1}\binom{7}{2}=25,137$$

or

$$\binom{10}{2}\binom{8}{2}\binom{7}{2}-\binom{9}{1}\binom{7}{1}\binom{7}{2}=25,137$$

Note that if there were n full professors, m associate professors, and r assistant professors, and there are k professors chosen from each rank, then this gives a weird identity. Both sides of the identity would have $\binom{r}{k}$ so I will divide through by that term.)

$$\binom{n-1}{k}\binom{m-1}{k}+\binom{n-1}{k-1}\binom{m-1}{k}+\binom{n-1}{k}\binom{m-1}{k-1}=\binom{n}{k}\binom{m}{k}-\binom{n-1}{k-1}\binom{m-1}{k-1}$$

5. On a piano, there are twelve notes per octave. A *scale* can be considered a selection of notes from an octave where you can ignore the order that the notes are selected. You can think of these notes as just the set $\{1, 2, \ldots, 12\}$. If we ignore the starting note for the scale, how many five-note scales are there? How many seven-note scales? (For musicians, ignoring the starting note is like considering A minor and C major to be the same scale because they have exactly the same notes).

Solution. This problem caused some confusion. Sorry about that. I was just trying to think of an interesting application. The answers are meant to be just $\binom{12}{7}=792$ and $\binom{12}{5}=792$. These two values are equal and it is interesting that the number of 7-note scales is the same as the number of 5-note scales (under a certain interpretation of scales).

6. How many solutions are there to

$$x+y+z=11$$

when x, y, z are integers and $x, y, z \geq 0$?

$$\binom{13}{2}$$

7. Use the binomial theorem to exampe $(x-2y)^6$.

$$\binom{6}{0}x^0(-2y)^6+\binom{6}{1}x^1(-2y)^5+\binom{6}{2}x^2(-2y)^4+\binom{6}{3}x^3(-2y)^3+\binom{6}{4}x^4(-2y)^2+\binom{6}{5}x^5(-2y)^1+\binom{6}{6}x^6(-2y)^0$$
$$=64y^6-192xy^5+240x^2y^4-160x^3y^3+60x^4y^2-12x^5y+x^6$$

8. (For graduate sutdents.) Expand the expression $(x+2y-z)^4$ using the multinomial theorem.

First, we'll enumerate the ways to make (n_1, n_2, n_3) with $n_1+n_2+n_3=4$. We know that the number of possibilities is $\binom{4+2}{2}=\binom{6}{2}=15$.

$$(0,0,4),(0,1,3),(0,2,2),(0,3,1),(0,4,0),(1,0,3),(1,1,2),(1,2,1),(1,3,0)$$
$$(2,0,2),(2,1,1),(2,2,0),(3,0,1),(3,1,0),(4,0,0)$$

Then the solution is

$$\begin{aligned}(x+2y-z)^4 &= \binom{4}{0,0,4}x^0(2y)^0(-z)^4+\binom{4}{0,1,3}x^0(2y)^1(-z)^3+\binom{4}{0,2,2}x^0(2y)^2(-z)^2\\
&+\binom{4}{0,3,1}x^0(2y)^3(-z)^1+\binom{4}{0,4,0}x^0(2y)^4(-z)^0+\binom{4}{1,0,3}x^1(2y)^0(-z)^3\\
&+\binom{4}{1,1,2}x^1(2y)^1(-z)^2+\binom{4}{1,2,1}x^1(2y)^2(-z)^1+\binom{4}{1,3,0}x^1(2y)^3(-z)^0\\
&+\binom{4}{2,0,2}x^2(2y)^0(-z)^2+\binom{4}{2,1,1}x^2(2y)^1(-z)^1+\binom{4}{2,2,0}x^2(2y)^2(-z)^0\\
&+\binom{4}{3,0,1}x^3(2y)^0(-z)^1+\binom{4}{3,1,0}x^3(2y)^1(-z)^0+\binom{4}{4,0,0}x^4(2y)^0(-z)^0\\
&=z^4-8yz^3+24y^2z^2-32y^3z+16y^4-4xz^3+24xyz^2-48xy^2z+32xy^3\\
&+6x^2z^2-24x^2yz+24x^2y^2-4x^3z+8x^3y+x^4\end{aligned}$$

I haven't attempted to order this polynomial in any particular way. There are different ways of ordering possible, for example, ordering by largest exponent on x, then on y, then on z, ordering by largest exponent for any variable, positive coefficients then negative coefficients, etc.