HW3, for MATH441, STAT461, STAT561, due September 20th

1. Suppose A and B are independent with $0 < P(A), P(B) < 1$. Show that A^c and B^c are also independent.

Solution. We know that $P(AB) = P(A)P(B)$ and that $P(A|B) = P(A)$. There will be lots of ways to show that A^c and B^c are independent. Here is one way.

It is sufficient to show that $P(A^c|B^c) = P(A^c)$.

$$\begin{aligned} P(A^c|B^c) &= 1 - P(A|B^c) \\ &= 1 - \frac{P(AB^c)}{P(B^c)} \\ &= 1 - \frac{P(A) - P(AB)}{1 - P(B)} \\ &= 1 - \frac{P(A) - P(A)P(B)}{1 - P(B)} \\ &= 1 - \frac{P(A)(1 - P(B))}{1 - P(B)} \\ &= 1 - P(A) \\ &= P(A^c) \end{aligned}$$

This is what was needed to be shown.

2. Lead and bacteria are two commmon sources of contamination in a water distribution system. Suppose 4% of the water distribution systems are contaminated by lead, and only 2% of the water distribution systems are contaminated by bacteria. Assume that the events of lead and bacterial contamination are statistically independent.

(a) Determine the probability that a water distribution system selected at random for inspection is contaminated.

(b) If a system is indeed contaminated, what is the probability that it is caused by lead only?

Solutions.

(a) $P(L \cup B) = P(L) + P(B) - P(LB) = (.04) + (.02) - (.04)(.02) = 0.0592$.

(b) $L \cup B$ is the event that there is contamination.

$$P(LB^c|L \cup B) = \frac{P(LB^c(L \cup B))}{P(L \cup B)} = \frac{P(LLB^c \cup LBB^c)}{P(L \cup B)} = \frac{P(LB^c)}{P(L \cup B)} = \frac{(.04)(.98)}{.0592} = 0.662$$

Here I used the facts that $LL = L$ and $BB^c = \emptyset$.

3. According to past records, a batch of mixed concrete supplied by a certain supplier can be of high quality (H), medium quality (M), or low quality. (L), with respective probabilities of 0.2, 0.7, and 0.1, respectively Suppose that the probability of failure of a reinforced concrete component would be 0.001, 0.01, or 0.1 depending whether the quality of the concrete is high (H), medium (M), or low (L), respectively.

(a) What would be the probability of failure of a reinforced concrete structural component cast with a batch of concrete supplied by the manufacturer?

(b) A test may be performed to give more information on the quality of concrete supplied by the manufacturer. The probabilities of passing the test for high, medium, and low quality concrete are 0.90, 0.70, and 0.20, respectively. If a batch of concrete passed the test
(i) What is the probability that it will be of high quality?
(ii) In this case, i.e., concrete passed the test, what would be the probability of failure of a structural component cast from this batch of concrete?

Solution.

$$P(F) = P(F|H)P(H) + P(F|M)P(M) + P(F|L)P(L) = 0.0172$$

(b) A test may be performed to give more information on the quality of concrete supplied by the manufacturer. The probabilities of passing the test for high, medium, and low quality concrete are 0.90, 0.70, and 0.20, respectively. If a batch of concrete passed the test
(i) What is the probability that it will be of high quality?
(ii) In this case, i.e., concrete passed the test, what would be the probability of failure of a structural component cast from this batch of concrete?

Solution. Let T be the event that the test is passed. The probability of passing the test is

$$P(T) = P(T|H)P(H) + P(T|M)P(M) + P(T|L)P(L) = (0.2)(0.9) + (0.7)(0.7) + (0.2)(0.1) = 0.69$$

$$\begin{aligned}
P(H|T) &= \frac{P(T|H)P(H)}{P(T)} = \frac{(0.9)(0.2)}{0.69} = 0.2608696 \\
P(M|T) &= \frac{P(T|M)P(M)}{P(T)} = \frac{(0.7)(0.7)}{0.69} = 0.7101449 \\
P(L|T) &= \frac{P(T|L)P(L)}{P(T)} = \frac{(0.2)(0.1)}{0.69} = 0.02898551
\end{aligned}$$

Note that $P(H|T) + P(M|T) + P(L|T) = 1$. (This is not a coincidence–the concrete must be one of the three categories the way the problem was set up.)

(ii)

$$\begin{aligned}
P(F|T) &= P(F|TH)P(H|T) + P(F|TM)P(M|T) + P(F|TL)P(L|T) \\
&= P(F|H)P(H|T) + P(F|M)P(M|T) + P(F|L)P(L|T) \\
&= (.001)(0.2608696) + (.01)(0.7101449) + (.1)(0.02898551) \\
&= 0.003634783
\end{aligned}$$

Here we assumed that passing the test is irrelevant if we know the quality of the concrete, so that $P(F|TH) = P(H)$. Note that this lower than $P(F)$ when not conditioning on passing the test, which makes sense. We can think of this calculation as redoing the calculation in (a) but using "updated" probabilities for H, M and L. In other words, we repeat the calculation for (a) using 0.2608696 instead of 0.2, 0.7101449 instead of 0.7, and 0.02898551 instead of 0.1. Given a passed test, the updated probability for high quality goes up, low quality goes down, and medium quality barely goes up.

4. Two players take turns throwing a basketball until someone makes a basket, who is called the winner. The probability that the first player makes a basket is 0.4. The probability that the second player makes a basket is 0.5. Assuming that the players take turns until someone makes a basket and that each throw is independent with probabilities not changing, what is the probability that the first player wins?

Solution. You can use the infinite series approach or conditional probabilities. We'll use conditional probability.

Let A be the event that the first player wins. Let A_1 be the event that the first player wins on the first basket, and let B_1 be the event that the second player gets the basket on his or her first try. Then we condition on what happens initially, only paying attention to events for which the first player can win:

$$\begin{aligned}
&P(A) = P(A|A_1)P(A_1) + P(A|A_1^c B_1^c)P(A_1^c B_1^c) \\
&P(A) = P(A|A_1)P(A_1) + P(A|A_1^c B_1^c)P(A_1^c)P(B_1^c) \\
&\Rightarrow P(A) = 1 \cdot (0.4) + P(A)(0.6)(0.5) \\
&\Rightarrow P(A)(1 - 0.3) = 0.4 \\
&\Rightarrow P(A) = 0.4/0.7 = 4/7 \approx 0.57
\end{aligned}$$

Therefore, the first player has an advantage even though the first player is not as good making a shot. (I.e., going first gives an advantage that more than compensates for the difference in skill.)

5. A bag contains 9 marbles, of which 3 are red and 6 are blue. Players 1 and 2 take turns drawing marbles from the bag without replacement until a red marble is drawn. (Assume all marbles in the bag are equally likely to be drawn each round.) For example, Player 1 draws a blue from the bag with 9 marbles, then Player 2 draws a blue from the bag with 8 marbles remaining, then Player 1 draws from the bag with 7 marbles remaining, etc. What is the probability that Player 1 is the first to draw the red marble?

Solution. Player 1 wins if the sequence of red and blue marble draws is one of the following

$$r, bbr, bbbbr, bbbbbbr$$

where, for example, *bbbbr* means that player 1 got blue, players 2 got blue, then player 1 got blue, player 2 got blue, and finally on the fifth draw, player 1 got red.

The desired probability is the sum of these cases

$$P(r) = 3/9 = \frac{28}{84}$$
$$P(bbr) = \frac{6}{9}\frac{5}{8}\frac{3}{7} = \frac{15}{84} = 0.1785714$$
$$P(bbbbr) = \frac{6}{9}\frac{5}{8}\frac{4}{7}\frac{3}{6}\frac{3}{6} = \frac{5}{84} = 0.05952381$$
$$P(bbbbbbr) = \frac{6}{9}\frac{5}{8}\frac{4}{7}\frac{3}{6}\frac{2}{5}\frac{1}{4}\frac{3}{3} = \frac{1}{84} = 0.01190476$$
$$P(\text{Player 1 gets red}) = \frac{28+15+5+1}{84} = \frac{49}{85} = 0.5833$$

6. (Graduate students). Suppose six fair, independent 6-sided are tossed. What is the probability of getting three pairs? Examples of tosses that result in three pairs include $(2,2,3,3,5,5), (2,6,6,4,4,2)$, etc. Assume that for the three pairs, there are three distinct values. I.e., $(2,2,3,3,3,3)$ does not count as three pairs.

Solution. There are $\binom{6}{3}$ ways to choose which 3 numbers will form the pairs. Once they are chosen you have a "word" with three characters each repeated once. This is like rearrangments of $ALBUQUERQUE$, and there are $\frac{6!}{2!2!2!}$ ways to rearrange these characters. The solution is

$$\frac{\binom{6}{3}\frac{6!}{2!2!2!}}{6^6} = 0.03858025$$

Here is some R code that simulates to check that your answer is at least close (to third decimal place or so). Using R like this isn't necessary for this course, but it is a way to check your answer if you are not sure. Without the `table()` command (which makes it easy but would be hard to think of if you weren't aware of it), you could first sort the values, then make sure that the second value is the same as the first, the third value is distinct from the second, the fourth value is the same as the third, etc.

```
I <- 100000

# indicator for whether event occurs or not initialized to all 0
event <- (1:I)*0

for(i in 1:I) {
dice <- sample(1:6,replace=T)

#if event occurs, put a 1 instead of 0
# table() lists the number of times each value occurs.
# each value occurs twice if and only if there are three pairs
if(max(table(dice))==2 & min(table(dice)==2)) event[i] <- 1
}

# simulated proportion of times you get three pairs
print(mean(event))
```

7. (Graduate students for both parts). (a) On my shelf I randomly place 10 books in row, two of which are probability textbooks. What is the probability that the two probability textbooks happen to be next to each other?

(a) A pearl necklace is made from n pearls, two of which are defective. What is the probability that the two defective pearls are next to each other? Here, assume that the order of the pearls is random (all orders are equally likely) and that the necklace has no first or last pearl. The pearls are just arranged on a circle.

Solutions. (a) The easiest way to think of this is to think of the two probability books as being stuck together (say with a rubber band or glue). Then you have 9 objects. There are 9! ways to arrange the 9 objects and two ways to decide which of the two probability books goes first. Thus the probability is

$$\frac{2 \times 9!}{10!} = \frac{2}{10} = \frac{1}{5}$$

(b) The easiest way to think of this one is to first imagine the first defective pearl is somewhere on the necklace (it doesn't matter where because there is no first pearl). Then how many ways can the second defective pearl be next to it. There are two pearls next to the first defective pearl, and there are $n-1$ positions where the second defective pearl could be, so the probability is $2/(n-1)$.

Another approach is to think of there being a first position and last position, but you count the pearls as being next to each other if one is in the first position and another is in the last position. The two pearls can be in $\binom{n}{2}$ possible positions (thinking of the defective pearls as unordered), and they are next to each other if they are in one of the following n positions:

$$(1,2),(2,3),(3,4),\ldots,(n-1,n),(n,n+1)$$

where $(n,n+1)$ is the same as $(1,n)$. There are n ways for them be adjacent out of $\binom{n}{2}$ possibilities, so the probability is

$$\frac{n}{\binom{n}{2}} = \frac{n}{n(n-1)/2} = \frac{2}{n-1}$$