HW6 DUE OCTOBER 26TH

1. The median of a continuous random variable is a value m such that $P(X \leq m) = 0.5$. This means that X has a 50% chance of being bigger than m and a 50% chance of being smaller than m. Consider the random variable X with density

$$f(x) = \begin{cases} kx^2 & 0 < x < 1 \\ 0 & \text{otherwise} \end{cases}$$

(a) Find the value k so that $f(x)$ is a density (i.e., integrates to 1).

Solution.

$$k \int_0^1 x^2 \, dx = kx^3/3 \Big|_0^1 = k/3 \Rightarrow k = 3$$

(b) Find the median m of X.

$$\int_0^m 3x^2 \, dx = 1/2 \Rightarrow x^3 \Big|_0^m = 1/2 \Rightarrow m^3 = 1/2 \Rightarrow m = 1/\sqrt[3]{2}$$

2. Suppose men from a certain population have mean height 69 inches with standard deviation 3 inches. Let X be the height of a randomly sampled man from this population.

(a) Find $P(X > 74 | X > 72)$

Solution. We suppose that the distribution is normal.

$$P(X > 74 | X > 72) = \frac{P(X > 74)}{P(X > 72)} = \frac{P[(X-69)/3 > (74-69)/3]}{P[(X-69)/3 > (72-69)/3]} = \frac{P(Z > 5/3)}{P(Z > 1)} = \frac{1 - \Phi(1.67)}{1 - \Phi(1)} = 0.3$$

(b) Find $P(X < 74 | X > 72)$

Given that $X > 72$, either $X > 74$ or $X < 74$, so the answer for (b) is 1 minus the answer for (a), so the answer is about 0.7.

(c) Find $P(67.5 < X < 68.49)$ (Although under a continuous distribution, the probability that someone is say, exactly 5 feet 8.000000...(with infinite precision) inches is 0, we can think of the probability that someone is 5'8" as the probability that their height is some number that would be rounded to 5'8".

$$\Phi((68.49 - 69)/3) - \Phi((67.5 - 69)/3) = 0.124$$

I think I intended this to be $P(68.48 < X < 68.5) = 0.001311099$.

3. Let Z be a standard normal random variable. I.e., Z has density

$$f(z) = \frac{1}{\sqrt{2\pi}} e^{-z^2/2}$$

Recall that $E[Z^2] = 1$ and use integration by parts to show that $E[Z^4] = 3E[Z^2] = 3$.

Solution. For the integration by parts, I'll use

$$u = z^3, du = 3z^2 dz, dv = -ze^{-z^2/2} dz, v = \int ze^{-z^2} \, dz = e^{-z^2/2}$$

$$\begin{aligned} E[Z^4] &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} z^4 e^{-z^2/2}\, dz \\ &= \frac{1}{\sqrt{2\pi}} \left[- \cdot z^4 e^{-z^2/2} \Big|_{-\infty}^{\infty} + 3 \int_{-\infty}^{\infty} z^2 e^{-z^2/2}\, dx \right] \\ &= 0 + 3E[Z^2] \\ &= 3 \end{aligned}$$

4. Let X come from a uniform (0,1) distribution. You can think of X as the point where a stick of unit length is broken at random. The two pieces of the stick then have lengths X and $1-X$. Find the probability that $\frac{X}{1-X}$ is less than $\frac{4}{5}$.

Solution.

$$P\left(\frac{X}{1-X} < \frac{4}{5}\right) = P\left(X < 4(1-X)/5\right) = P(X < 4/5 - 4X/5) = P(9X/5 < 4/5) = P(X < 4/9) = 4/9$$

5. Compute $P(X > 1)$ and $P(X > 2|X > 1)$ for the following distributions:

(a) Uniform (0,4). This random variable has density

$$f(x) = \begin{cases} 1/4 & 0 < x < 4 \\ 0 & \text{otherwise} \end{cases}$$

Solution. $P(X > 1) = 3/4$.

$$P(X > 2|X > 1) = P(X > 2)/P(X > 1) = (1/2)/(3/4) = 2/3.$$

(b) Exponential with mean 2 (rate 1/2)

Solution. $P(X > 1) = e^{-1/2} \approx 0.607$.

$$P(X > 2|X > 1) = P(X > 2)/P(X > 1) = e^{-(1/2)\cdot 2}/e^{-1/2} \approx 0.607$$

(c) Normal with mean 2, variance 1

$$P(X > 1) = 0.135, \quad P(X > 2|X > 1) = 0.512$$