Homework 8

1. Let X and Y be independent and identically distributed where X has density

$$f_X(x) = \frac{1}{x^2}\ I(x > 1)$$

Let $U = X/Y$, $V = X$. Find the joint density for (U, V). Also find the marginal density $f_U(u)$.

We have $x = v$ and $y = v/u$. Thus, for the Jacobian we have

$$\begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix} = \begin{vmatrix} 1/y & -1/y^2 \\ 1 & 0 \end{vmatrix} = 1/y^2 = u^2/v$$

For limits on u and v notice that u is equal to v divided by a number bigger than 1. Consequently, $1 < v < \infty$ and $0 < u < v$. There is no restriction on u with respect to 1 (u can be bigger or smaller than 1), so we have $v > \max\{u, 1\}$.

The joint density is

$$\begin{aligned} f_{U,V}(u, v) &= f_{X,Y}(x, y)|J|^{-1} \\ &= f_{X,Y}(v, v/u) \cdot v/u^2 I(0 < u < v < \infty, v > \max\{u, 1\}) \\ &= \frac{1}{v^2}\frac{u^2}{v^2}\frac{v}{u^2} I(0 < u < v < \infty, v > \max\{u, 1\}) \\ &= \frac{1}{v^3} I(0 < u < v < \infty, v > \max\{u, 1\}) \end{aligned}$$

The marginal density for U is

$$f_U(u) = \int_{\max\{u,1\}}^{\infty} f_{U,V}(u, v)\ dv$$

To work out the integral, we split it up into two cases depending on whether u is less than 1 or not. If u is less than 1, then $\max\{u, 1\} = 1$, so 1 is the lower limit. Otherwise, $\max\{u, 1\} = u$, so u is the lower limit. Thus, if $0 < u < 1$, then

$$f_U(u) = \int_1^{\infty} \frac{1}{v^3}\ dv = -\frac{1}{2}v^{-2}\Big|_1^{\infty} = \frac{1}{2}$$

If $u \geq 1$, then

$$f_U(u) = \int_u^{\infty} \frac{1}{v^3}\ dv = -\frac{1}{2}v^{-2}\Big|_u^{\infty} = \frac{1}{2u^2}$$

The density for U can be written

$$f_U(u) = \begin{cases} \frac{1}{2} & 0 < u < 1 \\ \frac{1}{2u^2} & u \geq 1 \\ 0 & \text{otherwise} \end{cases}$$

2. Let X and Y be independent and identically distributed exponential random variables with rate λ. Let $U = X/Y$ and let $V = XY$. Find the joint desnity for (U, V). Also find the marginal densities $f_U(u)$ and $f_V(v)$.

First, we have $uv = (x/y)xy \Rightarrow x^2 = uv \Rightarrow x = \sqrt{uv}$. Also $y = v/x \Rightarrow y = v/\sqrt{uv} = \sqrt{v/u}$. For the Jacobian, we get

$$\begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix} = \begin{vmatrix} 1/y & -x/y^2 \\ y & x \end{vmatrix} = 2x/y = 2u$$

The joint density is

$$\begin{aligned} f_{U,V}(u,v) &= f_{X,Y}(\sqrt{uv}, \sqrt{u/v})\frac{1}{2u} I(0 < u, v < \infty) \\ &= \lambda^2 e^{-\lambda(\sqrt{uv}+\sqrt{u/v})}\frac{1}{2u} I(0 < u, v < \infty) \\ &= \lambda^2 e^{-\lambda\sqrt{u}(\sqrt{v}+1/\sqrt{v})}\frac{1}{2u} I(0 < u, v < \infty) \end{aligned}$$

The marginals are

$$\begin{aligned} f_U(u) &= \int_0^\infty \lambda^2 e^{-\lambda\sqrt{u}(\sqrt{v}+1/\sqrt{v})}\frac{1}{2u}\, dv \\ f_V(v) &= \int_0^\infty \lambda^2 e^{-\lambda\sqrt{u}(\sqrt{v}+1/\sqrt{v})}\frac{1}{2u}\, du \end{aligned}$$

We'll leave it at that—this doesn't look tractable.

3. Let U_1 and U_2 be uniform(0,b), i.e., they have density

$$f(u) = \frac{1}{b}\ I(0 < u < b)$$

Find the density for $U = U_1 + U_2$. You can use a convolution to solve this or a bivariate transformation, or just by using the CDF method.

Solution. I'll use the convolution method. Recall that the density for a convolution $X+Y$ of two positive random variables is

$$f_{X+Y}(z) = \int_{-\infty}^{\infty} f_Y(y) f_X(z-y)\, dy$$

In this case, z can take on values from 0 to $2b$. If $z < b$, then in order for $z - u_2 > 0$, we need $u_2 < z$. If $z > b$, then we need $z - y < b \Rightarrow y > z - b$. We evaluate the convolution separately for $0 < z < b$ and $b < z < 2b$. For $0 < z < b$, we have

$$\begin{aligned} f_{U_1+U_2}(u) &= \int_0^z \frac{1}{b}\frac{1}{b}\, du_2 \\ &= \frac{u_2}{b^2}\bigg|_0^z = \frac{z}{b^2} \end{aligned}$$

For $b < z < 2b$, we have

$$\begin{aligned} f_{U_1+U_2}(u) &= \int_{z-b}^b \frac{1}{b^2}\, du_2 \\ &= \frac{u}{b^2}\bigg|_{z-b}^b \\ &= \frac{b-(z-b)}{b^2} \\ &= \frac{2b-z}{b^2} \end{aligned}$$

Thus,

$$f_U(u) = \begin{cases} \frac{z}{b^2} & 0 < z < b \\ \frac{2b-z}{b^2} & b \le z < 2b \\ 0 & \text{otherwise} \end{cases}$$

4. Let U be uniform(0,1) and let V be uniform(0,U).

(a) Find $E[V|U = u]$

(b) Find $Var(V|U = u)$

(c) Find $E[U]$

(d) Find $Var[U]$

Solutions.
(a) Given $U = u$, V is uniform$(0.u)$, which has a mean of $u/2$, so $E[V|U = u] = u/2$.

(b) Given $U = u$, V is uniform$(0, u)$ which as variance of $u^2/12$.

For (c) and (d) I'll give full credit for ($E[U]$ or $E[V]$) and for ($Var(U)$ or $Var(V)$). (c) Since U is uniform(0,1), the mean is 1/2. I meant to have asked $E[V]$, which is

$$E[V] = E\{E[V|U]\} = E[U/2] = \frac{1}{4}$$

(d) Since U is uniform(0,1), the variance is 1/12. I meant to have asked $Var[V]$. In this case, note that $E[U^2] = Var(U) + (E[U])^2 = 1/12 + 1/4 = 1/3$.

$$\begin{aligned} Var(V) &= E[Var(V|U)] + Var(E[V|U]) \\ &= E[U^2/12] + Var(U/2) \\ &= 1/36 + (1/4)(1/12) \\ &= 1/36 + 1/48 \\ &= (1/12)(1/3 + 1/4) \\ &= \frac{7}{144} \end{aligned}$$

(e)

$$f_{U,V}(u, v) = f_{V|U}(v|u)f_U(u) = (1/u)I(0 < v < u < 1)$$

(f) For $0 < v < 1$:

$$\begin{aligned} f_V(v) &= \int_v^1 f_{U,V}(u, v)\, du \\ &= \int_v^1 u^{-1}\, du \\ &= -\log u\Big|_v^1 \\ &= -\log v - \log 0 \\ &= -\log v \end{aligned}$$

Thus,

$$f_V(v) = -\log v I(0 < v < 1)$$

5. Considering rolling two 8-sided dice, where the two dice are independent. Let X be the value of the first die and Y the value of the sum of the two dice. Find the joint moment generating function of X and Y.

Solution. For a joint moment generating function for two variables, we have

$$M(t_1, t_2) = E[e^{t_1 x + t_2 y}] = \sum_{x=1}^{8} \sum_{y=2}^{16} e^{t_1 x + t_2 y} P(X = x, Y = y)$$

I'll just write out the first few terms, but it would be tedious to do the whole thing. Let Z be the value of the second die, so that $Y = X + Z$. Letting $P(i, j) = P(X = i, Y = j)$, we have $P(1, 2) = P(X = 1, Z = 1) = \frac{1}{64}$.

$P(1,3) = P(X = 1, Z = 2) = \frac{1}{64}$, etc. The probabilities in the sum are either 1/64, for combinations that are possible, or 0 for combinations that don't work, such as $P(X = 1, Y = 12)$. The moment generating function therefore looks something like this

$$M(t_1, t_2) = \frac{1}{64}\left(e^{t_1+2t_2} + e^{t_1+3t_2} + \cdots + e^{8t_1+16t_2}\right)$$