Math & Music

I

Chapter 1

Introduction to rhythm and the rhythmic pyramid

We begin with the idea of the rhythmic pyramid. The idea is that given any duration, we could either double it or split it into two equal halves, like cutting a slice of cake.

Much of the music we listen to in rock, blues, jazz, classical, and other traditions is based on phrases that are four beats long. A *whole note* has this duration of four beats. We also have

- A *half note* has a duration of half of a whole note, or two beats.
- A *quarter note* lasts for one-half of a half-note, or one beat.
- An *eighth note* lasts for half as long as a quarter note
- A *sixteenth note* last for half as long as an eighth note

In principal, there's no end to the number of subdivisions you can imagine — 64th notes, 128th notes, etc. We think of the whole note at the top of the pyramid. But we can work from the bottom to the top as well, by saying that a quarter note lasts twice as long as an eighth note, for example.

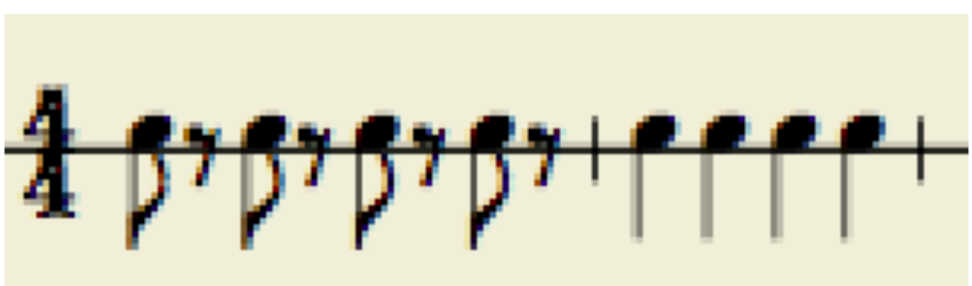
We can also say that a quarter note has the same duration as two eighth notes, an eighth note has the same duration as two sixteenth notes, and so forth.

We can write these relationships this way

$$
\begin{aligned}
\text{𝅝} &= \text{𝅗𝅥} + \text{𝅗𝅥} \\
\text{𝅗𝅥} &= \text{♩} + \text{♩} \\
\text{♩} &= \text{♪} + \text{♪} \\
\text{♪} &= \text{𝅘𝅥𝅯} + \text{𝅘𝅥𝅯} \\
\text{𝅘𝅥𝅯} &= \text{𝅘𝅥𝅰} + \text{𝅘𝅥𝅰}
\end{aligned} \tag{1.1}
$$

For the equations above, it is important to remember that this is intended to just mean that the durations are the same, not that they sound the same.

For percussion instruments such as drums, a note that is written as a duration such as a quarter note or a half note does not necessarily last that long. This can be true of a guitar as well when the sustain is not very long—the note might fade too fast to be heard for the entire written duration. As a result the rhyths in the following two measures of 4/4 time might sound identical when played on percusion instrument or a guitar with short sustain at a slow tempo.



We can express the same relationships as fractions

$$\begin{aligned} 1 &= \frac{1}{2} + \frac{1}{2} \\ \frac{1}{2} &= \frac{1}{4} + \frac{1}{4} \\ \frac{1}{4} &= \frac{1}{8} + \frac{1}{8} \\ \frac{1}{8} &= \frac{1}{16} + \frac{1}{16} \\ \frac{1}{16} &= \frac{1}{32} + \frac{1}{32} \end{aligned}$$

Another important rhythmic concept is that of *dotting* notes. A dotted note is equal to one-and-a-half, or 50%, longer duration than the note without the dot. For example, a dotted quarter note, ♩., is 50% longer than a quarter note, ♩.

We can make this more precise using the rhythmic pyramid. Since a quarter note is equivalent in duration to two eighth notes, a dotted quarter note is equivalent in duration to a quarter note plus an eighth notes, or to three eighth notes

♩. = ♩ + ♪
= ♪ + ♪ + ♪

Rhythmic durations can be added just as fractions are added in math. For example, the mathematical statement that

$$\begin{aligned} \frac{1}{4} + \frac{1}{8} &= \frac{2}{8} + \frac{1}{8} \\ &= \frac{3}{8} \end{aligned}$$

is similar to the musical statement that

♩ + ♪ = ♩.

Dotted versions of whole notes, half-notes, quarter notes, eighth notes, and so forth can all be used, and they all follow a similar rhythm pyramid as their undotted versions:

$$
\begin{aligned}
\text{𝅝.} &= \text{𝅗𝅥.} + \text{𝅗𝅥.} \\
\text{𝅗𝅥.} &= \text{♩.} + \text{♩.} \\
\text{♩.} &= \text{♪.} + \text{♪.} \\
\text{♪.} &= \text{𝅘𝅥𝅯.} + \text{𝅘𝅥𝅯.} \\
\text{𝅘𝅥𝅯.} &= \text{𝅘𝅥𝅰.} + \text{𝅘𝅥𝅰.}
\end{aligned}
$$

We can also write

$$
\begin{aligned}
\text{𝅝} &= \text{𝅗𝅥} + \text{𝅗𝅥} \\
&= \text{𝅗𝅥} + \text{♩} + \text{♩} \\
&= \text{𝅗𝅥} + \text{♩} + \text{♪} + \text{♪} \\
&= \text{𝅗𝅥} + \text{♩} + \text{♪} + \text{𝅘𝅥𝅯} + \text{𝅘𝅥𝅯} \\
&= \text{𝅗𝅥} + \text{♩} + \text{♪} + \text{𝅘𝅥𝅯} + \text{𝅘𝅥𝅰} + \cdots
\end{aligned}
$$

If we express this as fractions, we get

$$
\begin{aligned}
1 &= \frac{1}{2} + \frac{1}{2} \\
&= \frac{1}{2} + \frac{1}{4} + \frac{1}{4} \\
&= \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{8} \\
&= \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \frac{1}{16} \\
&= \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \frac{1}{32} + \cdots
\end{aligned}
$$

The sum continues infinitely. In mathematics, this is called an infinite series. In a course in Calculus, you might study this series from a mathematical point of view. From a musical point of view, you are chopping the remaining amount of time in a measure of 4/4 time into two equal pieces over and over and again, so you can imagine making the notes smaller and smaller and still just perfectly filling the measure.

1.0.1 Rhythmic pyramid with triplets

Instead of dividing a unit of time into two equal pieces, we could divide it into three equal pieces. This can occur when there are three beats per musical phrase, such as in a waltz. In this case, we'd say that each measure has 3 beats. Often we divide individual beats into three equal durations, making a triplet.

In triplet-based music, such as much of jazz and blues, however, divisions into three are often then subdivided into two equal, smaller durations. For example, a beat divided into triplets might be divided again by subdividing each triplet into two notes, creating sixteenth-note triplets. This divides a beat into 6 equal durations. If instead, we took a triplet and divided each note of the triplet into three notes, we'd get a subdivision of 1 beat into 9 equal durations. This is possible but less common. A more common case might be a jazz waltz, played with 3 beats per measure, where each beat is divided into triplets, creating 9 notes per measure. Subdividing these triplets into three notes each would create 27 ($= 9 \times 3$) notes per measure.

1.0.2 Rhythmic pyramid with rests

In addition to notes being played, silence can also last the same amount of time as notes being played.

W: 20 Jan 16

Here we explain relationships between whole notes, half notes, quarter notes, eighth notes, 16th notes, and 32nd notes. We also explain dotted notes, but did not explain double dotted notes. Also similar durations for rests. Discussed adding durations as adding fractions including the series

$$\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots = 1$$

I believe that "Clapping Music" by Steve Reich was discussed in this class.

F: 22 Jan 16

Similar to previous lecture but introduced triplets. Songs discussed included "America" from *West Side Story* and *Don't Tread on Me* by Metallica.

Chapter 2

Introduction to permutations

2.1 Circular permuations

Drummers often use the word *permutation* to describe certain types of variations. For example, if you play a paradiddle, meaning a sticking pattern of

$$RLRRLRLL$$

where R means playing with the right hand, and L means playing with the left, a variation on this exercise is the *invertedparadiddle*

$$RLLRLRRL$$

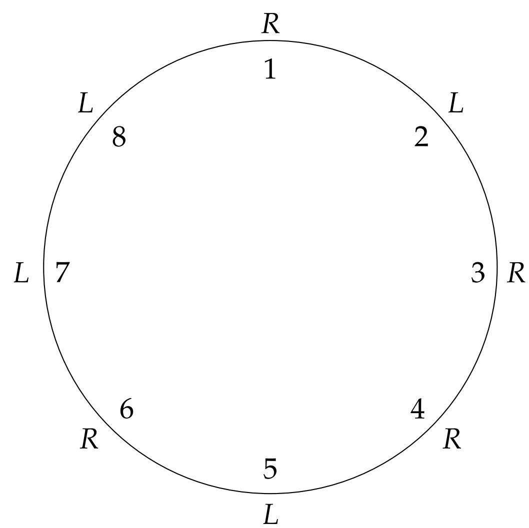
The inverted paradiddle is a permutation of the standard paradiddle. What does this mean?

The two patterns have a lot in common. Both involve playing four Rs and four Ls in a certain sequence, and in both cases, no hand plays more than two notes in a row. In mathematical lingo, a permuation of a sequence of objects (such as letters) would be any rearrangement of the objects into a new sequence. By this meaning of permuation,

$$LLLLRRRR$$

would count as a permutation of the paradiddle.

However, drummers often use *permutation* to mean something more specific, which in mathematical jargon is called a *circular permutation*. For this type of permuation, we imagine the sequence wrapped around a circle. We can start the sequence somewhere on the circle. Starting at a new place on the circle creates a circular permuation of the original sequence.



If we follow the circle at different starting positions, we get different paradiddles:

Starting position	paradiddle
1	*RLRR LRLL* (standard)
3	*RRLR LLRL* (reversed)
4	*RLRL LRLR* (delayed)
6	*RLLR LRRL* (inverted)

Starting at other positions results in left-handed paradiddles.. These paradiddle variations are all circular permutations of each other which we see by looking at the circle. We can also see that there are no other circular permutations of these paradiddles. For example, the pattern

$$RRLL\ RLRL$$

is not a circular permutation of the paradiddle because there is no way to encounter two *R*s followed by two *L*s anywhere on the circle.

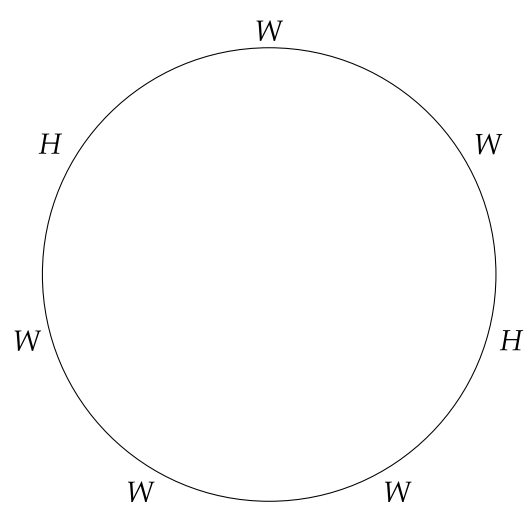
Another approach to thinking of paradiddles as circular permuations of each other is to imagine playing a paradiddle in a loop over and over again. This is similar to the idea of looping around the circle repeatedly. Then you can see paradiddle variations within the standard paradiddle. In the following table, the standard paradiddle is played twice, and variations are put in bold.

sticking	paradiddle variation
RL**RR** **LRLL** **RL**RR LRLL	reversed
RLR**R** **LRLL** **RLR**R LRLL	delayed
RLRR L**RLL** **RLRR** **L**RLL	inverted

This same concept can be used to illustrate how scales and modes are related to each other. If we consider a major scale, such as C major, the space between each note is either a whole step (equivalent to two frets on a guitar or two adjacent keys on a piano, white or black), or a half-step (equivalent to one step on a guitar or one key on a piano). The pattern is

$$WWHWWWH$$

where W means a whole step, and H means a half-step.



Similar to the idea of the paradiddle, we can imagine starting at any of the positions on the circle and moving clockwise one full rotation. Different starting points give us different *modes*. This is similar to the idea of playing all the white notes in an octave of a piano but starting on a different piano key. The following table shows the different modes that result from different starting positions:

Starting position	pattern	mode	styles
1	$WWHWWWH$	Ionian (major)	classical, punk
2	$WHWWWHW$	Dorian	Irish, rock
3	$HWWWHWW$	Phrygian	flamenco, metal
4	$WWWHWWH$	Lydian	
5	$WWHWWHW$	Mixolydian	Irish, rock, punk
6	$WHWWHWW$	Aelian (minor)	classical, metal
7	$HWWHWWW$	Locrian	metal

Different modes are more common in different genres. A rough generalization is that happier-sounding music will tend to have patterns with Hs near the end of the sequence, and sadder or darker music (especially metal) will tend to prefer modes with Hs near the beginning of the sequence. Suppose we score each mode by the sum of the positions of where H occurs. I'll call this the "Happiness Statistic". For example, for major, H occurs in positions 3 and 7, so we'll score it as 10. Mixolydian gets a score of $3 + 6 = 9$. Then we get the following ordering of the modes, which differs from the order of finding them on the circle:

You might notice that the genres of music seem a bit less jumbled up using this organization of the modes as well. This procedure suggests certain creative possibilities. For example, if H occurred in positions 4 and 6, then you'd get a score of 10, which would be the same as Ionian or major. Would this sound similar? Would it be as easy to write energetic-sounding punk riffs in such a scale as in the major scale? Does this correspond to some other scale we haven't encountered yet?

Table 2.1: Modes arranged by the Happiness Statistic

mode	Happiness Statistic	styles
Lydian	$4+7=11$	
Ionian	$3+7=10$	classical, punk
Mixolydian	$3+6=9$	Irish, rock, punk
Dorian	$2+6=8$	Irish, rock
Aeolian	$2+5=7$	classical, metal
Phrygian	$1+5=6$	flamenco, metal
Locrian	$1+4=5$	metal

2.2 Permutations not on a circle

In the concert movie *Stop Making Sense* by the band Talking Heads, first one member of the band, David Byrne, performs a song by himself, singing and playing guitar. Then the bass player, Tina Weymouth, comes out and they play a song with just two performers. For the third song, there are three performers, and for the fourth song there are four performers. Assuming that the band members could have come out in any order, how many ways could the band members have come out one at a time? We'll answer this question in a few paragraphs.

As another application, when a band records an album, they have to decide on the sequence of tracks that will appear on the CD or track listing. If an album has 10 tracks, how many sequences are possible?

There are so many possibilities that it isn't feasible to write them all down. Instead, let's consider a smaller example first. Suppose there only three tracks, call *Air*, *Bat*, and *Cat*. To decide on an order of the tracks, we can list all sequences:

1. Air, Bat, Cat
2. Air, Cat, Bat
3. Bat, Air, Cat
4. Bat, Cat, Air
5. Cat, Air, Bat
6. Cat, Bat, Air

There are six possibilities. But how could we think about this problem more systematically to deal with larger examples?

One way of thinking about it is that there were three choices for the first song. Then once the first song was chosen, there were two choices for the second song, *for each of the three choices of the first song*. This leads to $3 \times 2 = 6$ choices for the first two songs. Once the first two songs are chosen, the third is forced to be last, but we could think of this as $3 \times 2 \times 1 = 6$.

For the Talking Heads concert, there are four band members in the main band (there are also other hired musicians). From the main four members, there were four choices for the first member to appear, three for the second member, two for third, and one choice remaining for the last member. So the total number of sequences possible was

$$4 \times 3 \times 2 \times 1 = 24.$$

For the example of the album with 10 tracks, the number of possible track listings is

$$10 \times 9 \times 8 \times 7 \times 6 \times 5 \times 4 \times 3 \times 2 \times 1 = 3,628,800$$

These types of counting problems arise often enough that they are given a special notation, call *factorials*. We read $n!$ as "n factorial. We can think of $n!$ in several ways, depending on what is most convenient:

$$\begin{aligned}
n! &= 1 \times 2 \times \cdots \times n \\
n! &= n \times (n-1) \times (n-2) \times \cdots 2 \times 1 \\
n! &= n \times (n-1)!
\end{aligned}$$

Here are some factorials for small numbers

$$
\begin{aligned}
0! &= 1 \\
1! &= 1 \\
2! &= 2 \\
3! &= 6 \\
4! &= 24 \\
5! &= 120 \\
6! &= 720 \\
7! &= 5,040 \\
8! &= 40,320 \\
9! &= 362,880 \\
10! &= 3,628,800 \\
11! &= 39,916,800 \\
12! &= 479,001,600 \\
20! &= 2.4 \times 10^{18} \\
30! &= 2.7 \times 10^{32} \\
40! &= 8.2 \times 10^{47} \\
50! &= 3.0 \times 10^{64} \\
60! &= 8.3 \times 10^{81}
\end{aligned}
$$

The number 60! is similar to the number of atoms in the universe. If you have a band in your iPOD or iPHONE or other device with 60 songs, and you listen to all of them on shuffle play twice in a row, the chance that you get them all in the same order twice in a row is astronomically small. Winning the Powerball lottery with one ticket would be more likely than using shuffle play on a single album with 12 songs and getting the songs in the same order as the album.

One theme for this course is that possibilities are vast. The number of possible ways of creating music is much larger than what can actually be done. This means that there is much room for creativity.

As another application of permutations, some 20th-century composers such as Arnold Schoenberg had the idea that instead of using traditional musical scales, one could rearrange the 12 notes in an octave in a particular order, called a *tone row*, and play the notes in that order. It is up to the composer to choose the order of the notes, but they are often chosen so as not to emphasize any particular note and to not sound as if the music is written in a particular key. This is a crucial goal for *atonal* music. However, tone-rows could conceivably be chosen with other musical goals in mind. The tone-row concept has occasionally been applied by rock musicians as well, particularly by Ron Jarzombek (Blotted Science).

A natural question to ask is: How many tone rows are possible? Since there are 12 notes possible for the first note, 11 for the second, 10 for the third, and so on, the answer is 12!, nearly half of a billion. Arguably, the first note chosen matters little compared to the intervals between notes. That is, a tone-row created by transposing each note of another tone-row by the same amount (say, one fret on the guitar) will sound very similar. Ignoring the first note, there are 11!, or nearly 40 million tone-rows possible.

We will encounter the idea of tone-rows again later when we talk about ways of transforming music.

2.3 Partial permutations

For a partial permutation, we select some subset of objects from a set, and paying attention to the order in which the objects are selected. For example, if you have time to listen to three tracks from an album that has 10 songs, you could listen to tracks 1, 2, and 3. Or you could listen to tracks 3, 1, and 2, in that order, and that would be a different listening experience. Or you could listen to tracks 3, 10, and 5, and so on.

Partial permutations work the same way as regular permutations, except that instead of multiplying the possibilities starting at n and working all the way down to 1, we stop somewhere between n and 1.

For the example of listening to three tracks out of 10, there are 10 choices available for the first track, 9 remaining for the second track, and 8 for the third track. Thus, the number of ways of listening to 3 songs out of the 10 is

$$10 \times 9 \times 8 = 720$$

assuming that the order of the songs matters.

A special notation for partial permuations is $P(n,k)$ where n is the number of choices available, and k is the number of distinct choices made. A formula is

$$P(n,k) = \frac{n!}{(n-k)!} \tag{2.1}$$

For selecting three items from 10, where the order matters, the formula is

$$P(10,3) = \frac{10!}{7!} = \frac{10 \times 9 \times 8 \times 7 \times 6 \times 5 \times 4 \times 3 \times 2 \times 1}{7 \times 6 \times 5 \times 4 \times 3 \times 2 \times 1} = 10 \times 9 \times 8$$

There is a lot of cancellation in the numerator in the denominator, and instead of using the formula it is easier to think of $P(n,k)$ as having k terms (i.e., k pieces being multiplied). For example,

$$P(n-3) = n \times (n-1) \times (n-2).$$

For $P(n,k)$, the last term being multiplied is $(n-(k-1)) = (n-k+1)$. As a formula, this is

$$\begin{aligned} P(n,k) &= n \times (n-1) \times \cdots \times (n-k+1) \\ &= (n-0) \times (n-1) \times \cdots \times (n-(k-1)). \end{aligned}$$

There are k terms because there are k numbers in the list $0, 1, \ldots, k-1$. However, the form in equation (2.1) is more compact. Which way is easier to use can depend on the application.

M: 25 Jan 16

Types of permutations included circular as applied to drumming, including paradiddles (standard, inverted, delayed, reversed), and the idea that permutations include more than circular permutations. Spent a lot of time counting license plates, with and without distinct letters/numbers.

W: 27 Jan 16

More on permutations, including the formula $P(n,k) = n!/(n-k)!$. Discussed tone-rows and played some Ron Jarzombek doing a tone-row.

F: 29 Jan 16

Scales and modes as permuations, discussing $WWHWWWH$ as the major scale. Also Jazz scales as circular permutations of ascending melodic minor: $WHWWWWH$. Played some scales on a guitar in class.

Chapter 3

Introduction to combinations

3.1 Combinations versus permutations

If you listen to the song "I Would For You" by Jane's Addiction (easily found on Youtube.com), do you notice anything usual about this song by a rock band?

The song starts with just bass and vocals. Guitars never enter the song, even though this is a very guitar-oriented band, and then there are some light synthesizer sounds. Starting a song with just bass and vocals, and not having any drums or guitar, is an unusual *combination* of instruments for a rock band.

Most rock bands have essentially the same instrumentation in every song: guitar, drums, bass, and vocals, maybe keyboards of some kind. For parts of a song, especially an introduction, only a subset of the instruments might play. If there are four instruments in a band, say bass, drums, guitars, and vocals (counting vocals as an instrument), how many ways can only two instruments be playing? How do you list them all? What if there were 6 instruments and three of them were playing? Now how many combinations are there? We'll be answering this type of question in this section.

We use the concept of *combinations* in this setting rather than permutations because the order doesn't matter – bass and vocals are the same two instruments and vocals and bass.

Combinations are similar to partial permutations — we are interested in selecting some subset from a larger set — except that order doesn't matter.

For an everyday example outside of music, if you order a two-topping pizza, ordering a pepperoni and mushroom pizza is the same as ordering a mushroom and pepperoni pizza. The order in which you list the ingredients doesn't matter when you request the pizza. There might be a difference in how the pizza is made – probably one ingredient is put on the pizza before the other, and the order might affect the taste, but this is a decision you leave to the restaurant, and you wouldn't normally request the order in which they place the ingredients.

Combinations and permutations are both useful concepts in both mathematics and music. Which one is more useful depends on the question you have.

M: 1 Feb 16

Introduction to Combinations. Counting pizza toppings. Relationship to permuations,

$$C(n,k) = P(n,k)/k! = \frac{n!}{k!(n-k)!}$$

Jane's Addictions "I would for you" (on the live album) used an example of an unusual combination: bass and vocals (primarily, with slight synth).

W: 3 Feb 16

More on combinations. Binomial theorem, Pascal's triangle, Binomial expansion of $2^n = (1+1)^n$.

$$2^n = (1+1)^n = C(n,0) + C(n,1) + \cdots + C(n,n)$$

F: 5 Feb 16

Ideas from Benny Greb's Language of Drumming on diddling in various places: 16 ways to diddle four sixteenth notes and how to think of this as either 2^4 or $C(4,0) + C(4,1) + C(4,2) + C(4,3) + C(4,4)$. Discussed making variations on "Mary Had a Little Lamb" by diddling some of the notes and played MIDI examples in class. Also played part of "Some of My Favorite Things" by Coltrane.

Chapter 4

Graphs and Music

M: 8 Feb 16

Sick that day.

W: 10 Feb 16

Finished combinations, showed Chris Coleman Gospel Chops Youtube video describing six hand-foot combinations: HHKK, HKHK, HKKH, KKHH, KHKH, KHHK. Described how to count if you distinguish left from right: LRKK, RLKK, etc. as $\frac{4!}{2!1!1!}$ ways. Discussed number of rearrangements of *ALBUQUERQUE* and *MISSISSIPPI*

Discussed how similar musical staff notation is to Cartesian coordinate graphs.

F: 12 Feb 16

There is another sense of a graph that is important in mathematics, which is a set of vertices (also called nodes) and edges. Geometric objects such as triangles, squares, and polygons are often thought of in terms of their vertices and edges rather than in terms of their coordinates in a Cartesian graph.

Graphs in this sense have taken on a new importance in modern life as they are used to represent things like connections between people in

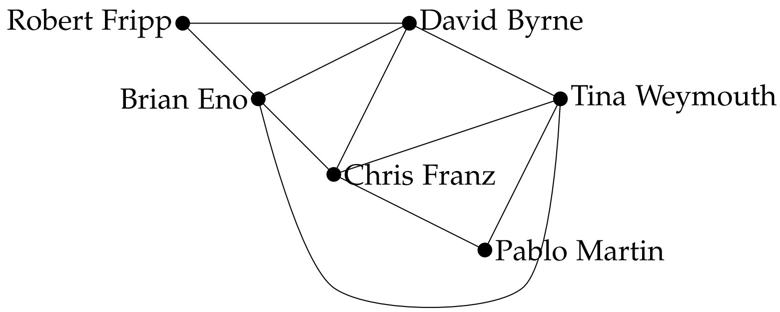


Figure 4.1: A graph depicting relationships between musicians.

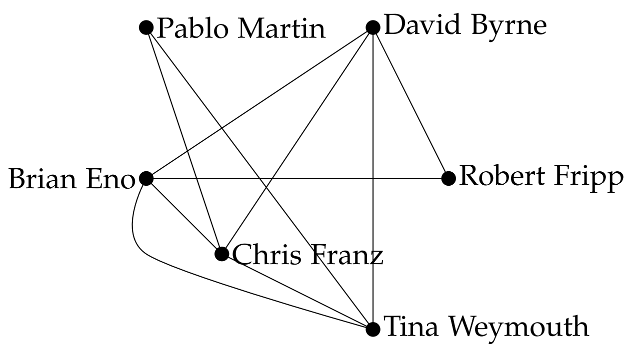
social media accounts, links between websites, and purchasing behavior of customers online (such as when a website says that customers viewing this item also viewed such-and-such a product....). Slightly more traditional uses of such graphs are describe shipping routes such as for a railroad network or airplanes, or to describe telephone connections for calling long distance.

Figure 4.1 is an example depicting a set of musicians. Each node represents a musician, and an edge is drawn if the two musicians have been in the same band or played on at least one album together. As an example, Brian Eno has played with King Crimson and Talking Heads, Tina Weymouth has played in Talking Heads and Tom Tom Club, and Pablo Martin has played in Tom Tom Club. In the diagram, nodes are represented black circles, and are labeled by musicians' names.

The game Six Degrees of Kevin Bacon (REF) illustrates this idea as well, where a graph could have actors as nodes, and an edge means that the actors have appeared in the same movie. The idea is that many actors can be connected within 6 edges from themselves to Kevin Bacon.

From the mathematical point of view, for this type of graph, the locations of the nodes don't matter at all. The nodes and edges for David Byrne, Chris Franz, Pablo Martin, and Tina Weymouth happen to form an irregular quadrilateral (a four-sided polygon), but could have been drawn as a square instead, or even with lines crossing. The lengths of the edges also don't usually matter, or whether edges are drawn straight or curved.

Curved edges might be drawn to make it easier for edges to not cross in the drawing. The following graph depicts exactly the same relationships and so is equivalent to the first one, in spite of lines crossing. This second graph is probably harder to read.



Problems that arise for such graphs include determining the minimum distance from one point to another, the minimum cost from one point to another (which might or might not have the minimum distance), determining nodes can be arranged so that lines don't cross, and predicting which nodes will grow more connections to other nodes (when graphs change over time).

Such graphs can be used in music in different ways. For example, vertices could be used to represent modes of a scale, and two vertices (i.e., two modes) could be connected if the two modes differ by at most one note. This is a useful way to visualize how similar to modes are in terms of how they sound. In particular, for this example, we'll consider modes that start on the same note. We'll compare C ionian (major), C dorian, C phyrgian, C lydian, C mixolydian, C aeolian, and C locrian. Are all of these modes connected by this definition of the graph?

It turns out that two of the seven diatonic modes (Ionian, Dorian, Phrygian, Lydian, Mixolydian, Aeolian, and Locrian) starting on the same note differ by one note if they are adjacent to each other in Table 2.1, which ordered modes by their Happiness Statistic.

We can make a graph of the modes considering two modes to be connected if they differ by exactly one note. Here we show the graph two

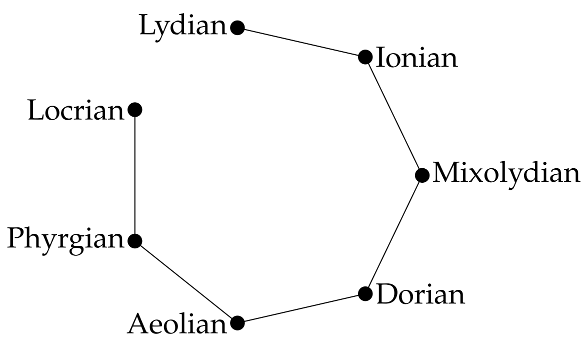


Figure 4.2: Graph of modal relationships. Edges indicate that modes starting on C differ by exactly one note.

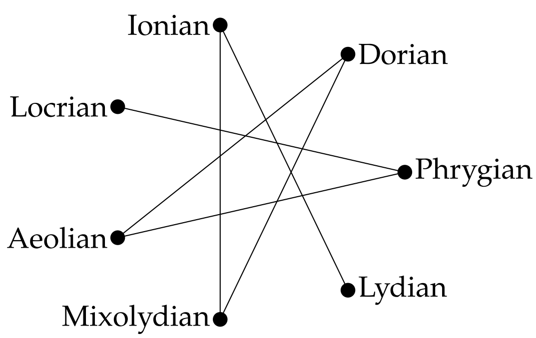


Figure 4.3: Graph of modal relationships. Edges indicate that modes starting on C differ by exactly one note.

ways, ordering the nodes in a circle either based on the Happiness Statistic, or based on the circle showing the distances $WWHWWWH$.

If we draw the modes on a circle in the order in which they are normally presented, based on playing seven white keys in a row first starting on C, then starting on D, etc., then the relationships would be graphed as in Figure 4.3

Both figures look a bit incomplete — the near circular Figure 4.2 needs one edge to complete a loop, and Figure 4.3 needs one edge to complete a seven-pointed star. In both cases apparently missing edge would connect the Lydian and Locrian modes. Is there a connection between these two modes? if you were willing to sharp the first note of C Lydian (so play C Locrian except play C-sharp instead C), you end up with C-sharp Locrian. Similarly, if you flat first note of C Locrian, this is sounds like B Lydian. It's almost as if by flatting the first scale degree, the modified Locrian suddenly becomes happy.

In addition to determining which modes are the most closely related, we can represent which key signatures are most closely related using a graph. Here we describe 12 keys in terms of the numbers of sharps and flats they have:

Table 4.1: Circle of Fifths

Key	sharps/ flats	notes						
A-flat	4	A-flat	B-flat	C	D-flat	E-flat	F	G
E-flat	3	A-flat	B-flat	C	D	E-flat	F	G
B-flat	2	A	B-flat	C	D	E-flat	F	G
F	1	A	B-flat	C	D	E	F	G
C	0	A	B	C	D	E	F	G
G	1	A	B	C	D	E	F#	G
D	2	A	B	C#	D	E	F#	G
A	3	A	B	C#	D	E	F#	G#
E	4	A	B	C#	D#	E	F#	G#
B	5	A#	B	C#	D#	E	F#	G#
F#	6	A#	B	C#	D#	E#	F#	G#
C#	7	A#	B#	C#	D#	E#	F#	G#

Often the sequence in the first column of Table 4.1 is memorized by musicians as the "Circle of Fifths" or "Circle of Fourths" (you increase by fifths going down the column, and increase by fourths going up the column). There are different rules that you can memorize to figure out what sharps or flats are in a key. The phrase "Circle of Fifths" suggests that a graph approach might also be useful.

Key signatures that are closely related will have similar numbers of sharps or flats. Something to be careful of, however, is that some notes can be written two different ways, e.g. as A# or B-flat. It turns out that because of this, C# and A-flat only differ by one note. If we re-write the key of C# using flats, we can write it as the key of D-flat, which has the notes

A-flat B-flat C D-flat E-flat F G-flat

and this disagrees with A-flat in only one note. Consequently, the graph of closeness for keys ends up being a closed circle, unlike the graph for the modes.

Understanding which keys are most closely related is useful for composers who might want either a smooth transition between sections, or a more dramatic change in the song.

Chapter 5

Counting rhythms with the Fibonacci sequence

M: Feb 15

Suppose you have rhythmic elements of length 1 and 2, such as eighth notes and quarter notes (quarter notes are twice long as eighth notes). If you play sequences of quarter notes and eighth notes, how many ways can you play them so that they add up to a certain length?

We can figure this out by listing all possibilities for small examples. If the total length is one eighth note, you can only play one eighth note. If the length is two eighth notes, then you can either play, one quarter or two eighth notes, so there are two possibilities. Here is a list of some possibilities:

The pattern here is that the number of ways matches something called the Fibonacci sequence. For this sequence, then nth number of the se-

length	number of ways	rhythms	
1	1	♪	*S*
2	2	♪♪ ♩	*SS*, *L*
3	3	♪♪♪ ♩♪ ♪♩	*SSS*, *SL*, *LS*
4	5		*SSSS SSL SLS LSS LL*
5	8		*SSSSS SSSL SSLS SLSS*
			LSSS LLS LSL SLL

quence is the sum of the previous two values:

$$F_n = F_{n-1} + F_{n-2}$$

for $n > 2$. To get the sequence started we use $F_1 = F_2 = 1$. This generates the sequence

$$\begin{aligned}
F_1 &= 1 \\
F_2 &= 1 \\
F_3 &= F_1 + F_2 = 2 \\
F_4 &= F_3 + F_2 = 3 \\
F_5 &= F_4 + F_3 = 5 \\
F_6 &= F_5 + F_6 = 8 \\
F_7 &= F_6 + F_5 = 13 \\
F_8 &= F_7 + F_6 = 21 \\
&\vdots
\end{aligned}$$

If the total length is n 8th notes, then there are F_{n+1} ways to arrange the quarter notes and eighth notes to fill up the space the exact amount. Because the indexing is off by one, we'll use S_n to mean the number of ways to arrange quarter notes and eighth notes to add up to n eighth notes. This means that $S_n = F_{n+1}$ for each n. For example, $S_5 = F_6$ is the number of ways to arrange quarter notes and eighth notes to add up to 5 eighth notes duration.

From the Fibonacci sequence, there are $S_7 = 21$ ways to arrange quarter notes and eighth notes to add up to 7 eighth notes. There are 34 ways to arrange quarter notes and eighth notes to add up to 4/4 measure with 8 8th notes.

Why does the Fibonacci sequence show up here?

One way of thinking about it is that the very first note is either long or short, either a quarter note or an eighth note. For example, if there are 8 8th notes to fill, then the first note is either a quarter note or an eighth note. If the first note is eighth note, then there are seven spaces remaining to be filled, which can occur in S_7 ways. If there first note is a quarter note, then there are six spaces remaining to be filled, which can occur in $S_6 = F_7$ ways. Therefore the number of ways to fill up 8 8th notes is

$$S_8 = S_7 + S_6$$

More generally, if there are n notes to fill, this can be done in S_n ways. If the first note is an 8th note, then there are S_{n-1} ways to fill the remaining space, and if the first note is an eighth note, then there are S_{n-2} ways to fill the remaining space. This means $S_n = S_{n-1} + S_{n-2}$.

There are many remarkable properties of the Fibonacci sequence. The approach given for generating the sequence allows you generate as many terms as you wish, but it is tedious to calculate, say F_{20}. A faster formula for calculating larger Fibonacci numbers is

$$F_n = \frac{(1+\sqrt{5})^n - (1-\sqrt{5})^n}{\sqrt{5}}$$

The formula looks messy becuase of the square roots, but when you evaluate it for particular values of n, then the square roots cancel out and the answer is an integer.

The value $\frac{1+\sqrt{5}}{2} \approx 1.618$ is called the *Golden Ratio* and has been known since ancient Greece, more than 2000 years ago. It is often used in architecture to form ratios for things like the height of a door to its width. The Golden Ratio is also related to the Fibonacci sequence in terms of the ratio of successive values of the sequence:

$$\frac{F_{n+1}}{F_n} \approx \frac{1+\sqrt{5}}{2}$$

For example

$$
\begin{aligned}
F_2/F_1 &= 1/1 = 1 \\
F_3/F_2 &= 2/1 = 2 \\
F_4/F_3 &= 3/2 = 1.5 \\
F_5/F_4 &= 5/3 = 1.67 \\
F_6/F_5 &= 8/5 = 1.6 \\
F_7/F_6 &= 13/8 = 1.625 \\
F_8/F_7 &= 21/13 = 1.615
\end{aligned}
$$

The approximation gets better for larger values of n.

W: Feb 17

More on the Fibonacci sequence.

F: Feb 17

Quiz and lots of YouTube examples illustrating odd time signatures or odd rhythmic groupings. Songs played in class:

- “Take Five”, Dave Brubeck: $3+3+2+2=10$ or $(1+2)+(1+2)+2+2=10$, similar to “Mission Impossible” theme
- “Larks Tongues in Aspic part 2” by King Crimson $3+3+2+2$ and variations
- “At Fate’s Hands” by Fates Warning, $3+3+2+3=11$ alternating with $3+2+3+3$, using permutations of the Take Five rhythm.
- “Stairway to Heaven” by Led Zeppelin, section right after the guitar solo, $3+3+3+3+2+2$ or $(2+1)+(2+1)+(2+1)+(2+1)+2+2$
- “A Pleasant Shade of Grey part XII” by Fates Warning, $5+7$
- “Schism” by Tool, $5+7$

- "Flesh and the Power It Holds" by Death, guitar solo section, $5+7$
- "Jambi" by Tool, $4+5$ for the guitar with every three notes played by the kick drum

Chapter 6

Octave Equivalence

Relations

Often in mathematics we are interested in comparing two objects at a time. Two objects might stand in different relations to each other, for example, two triangles might be equivalent in some sense (the shape or the same size or the same perimeter). Or for two numbers, the first number might be smaller than the second number.

A notation for describing relationships is aRb, where R is the relation, and a and b are the objects being compared. Although the objects are often mathematical, such as numbers, shapes, functions, and so on, they can be anything. They could people, words, musical pitches, chords, or key signatures.

To give some examples, let aRb mean that a is the father of b. Then the following are true

- dRl, where d is Darth Vader and l is Luke Skywalker
- aRb, where a is John Adams (the second US President) and b is John Quincy Adams (the 6th US President)

For the Darth Vader example, dRl is true but lRd is false. Generally, for the relation of fatherhood, it seems that if aRb is true (for any choice of

a and b), then bRa must be false.

However, consider the relation R where aRb means that a and b are married. In this case, if a is married to b, it seems that b must be married to a. In such a case we say that the relation R is *symmetric*. Some relations are symmetric, and some aren't.

Another example of a relationship is for numbers, say, or amounts of money. Let aRb if $a \geq b$. If $a = b$, then aRb and bRa are both true; however, this relationship does not count as symmetric because there are examples where aRb is true but bRa is false. For example, if $a = 3$ and $b = 2$, then aRb is true but bRa is false. If $a = 3$ and $b = 3$, then aRb and bRa are both true. A relation is called symmetric only if it is true for all choices of a and b that are relevant (in this case, we are assuming that a and b are numbers).

For this example, is it true that aRa? Yes, because $a \geq a$ for any number a. A relation with this property is called *reflexive*.

A final property that we are interested in is *transitivity*. A relation R is transitive if for any three objects, a, b, and c, if aRb and bRc, then aRc. The relation of being greater than or equal to is also transitive. That is, if $a \geq b$ and $b \geq c$, then $a \geq c$. Notice that the greater than or equal to sign follows the syntax of aRb with the two objects to either side of the relation. We just replace R with $\geq$. The greater than or equal to relation is reflexive and transitive, but is not symmetric.

Consider the set of all people who are either left-handed or right-handed but not both (not ambidextrous). We assume that each person in this set is either left-handed or right-handed. Now suppose aRb means that a and b both have the same handedness —either they are both left-handed or both right-handed. Then for any choice of a, b, and c,

- if aRb then bRa (symmetry)
- aRa (reflexivity)
- if aRb and bRc then aRc (transitivity)

The second property just means that you have the same handedness as yourself, which might seem a bit silly. If it isn't clear whether the third case is true, consider the following. Suppose aRb and bRc are true. Either b is left-handed or right-handed. Suppose b is right-handed. Then it must be the case that a is right-handed and that c is right-handed. So a and c have the same handedness, and aRc is true. If b is left-handed, then a is left-handed, and c is left-handed, so again aRc is true. Either way aRc is true.

A relation that has all three properties – symmetry, reflexivity, and transitivity — is special enough to get its own name, called an *equivalence relation*. For an equivalence relation R, if aRb is true, then a and b belong to the same *equivalence class*, and they are in some sense equivalent. This equivalence does not mean the same thing as equality. For the handedness example, R divides non-ambidextrous people into two sets (or classes): left-handed and right-handed. From the point of view of R each left-handed person is "equivalent" and each right-handed person is "equivalent".

Another example of equivalence classes would be triangles that are similar. In this case, two triangles a and b satisfy aRb if a and b have the same set of angles, regardless of how big the triangles are, where they are located, and so forth.

A musical application of the concept of equivalence classes is octaves. Two notes that are separated by one, two, or some number of octaves in some sense sound like the "same" note. Considering x and y to be two musical notes from $\{a, b, c, d, e, f, g\}$, we might write xRy if x and y are the "same" note, meaning they could both be middle C, or one could be middle C, and the other the C two octaves higher. We might also mean that x middle C played on a piano while y is the same C played on a trumpet, but in some sense they are playing the same note.

Rather than referring to these as equivalence classes, in music theory they are often called *pitch classes*. There are 12 pitch classes in standard tuning corresponding to the notes a,a-sharp, b, c, c-sharp, d, d-sharp, e, f, f-sharp, g, and g-sharp. Here we consider a-sharp and b-flat, for example, to be equivalent.

From this point of view, all notes with the value c are equivalent, all notes with the value c-sharp are equivalent, etc.

Chapter 7

Modular Arithmetic

In modular arithmetic, we imagine a number line that wraps around on itself like a clock. On a clock, if you are at 10:00, then four hours later it is 2:00. It is tempting to write this is as $10 + 4 = 2$, but writing it this way looks a little funny.

Instead of writing it this way, it is often writen as

$$10 + 4 \equiv 2 \pmod{12}$$

We read this as 10 plus 4 is congruent to 2 mod 12. In modular arithmetic, we identify the largest number with 0, much like in military or 24-hr time on a clock, midnight is 0:00 instead of 24:00.

One way of doing modular arithmetic, is to add numbers normally, but if you get a number at least as large as the modulus (12 for the clock example), subtract 12 from the result. Thus

$$10 + 4 = 14 \equiv 14 - 12 \equiv 2 \pmod{12}$$

Chapter 8

Symmetry

Symmetric scales

Symmetry in functions

Standard examples of functions that you would be exposed to in a first algebra class are things like

$$f(x) = x + 1, \quad f(x) = x^2, \quad f(x) = |x| + 3$$

Sometimes when graph a function such as a parabola, it has the property that the right side (to the right of the y-axis) looks like the mirror image of the left side. This is similar to the Retrograde opertion in music which is the mirror image of a melody or sequence of notes. Functions with this property are called *even functions*.

Sometimes functions look like they are symmetric through the origin, meaning that if you reflect the right hand-side across the y-axis, and then reflect again across the x-axis, the function in the top-right of the Cartesian plane looks like the function in the bottom left. This is similar to the Retrograde Inversion operation on melodies. Functions with this type of symmetry are called *odd functions*. Functions can be even, odd, or neither even nor odd.

There is a test for whether a function is even or odd. To check for even or odd functions, here are the rules:

- if for all x, $f(-x) = f(x)$, then the function is even
- if for all x, $f(-x) = -f(x)$, then the function is odd

As an example, let's check that $f(x) = 3x^2$ satisfies the property of an even function for a particular value of x, say $x = 1$. We need to check that $f(1) = f(-1)$. We find

$$\begin{aligned} f(-1) &= 3 \cdot (-1)^2 = 3 \cdot 1 = 3 \\ f(1) &= 3 \cdot (1)^2 = 3 \cdot 1 = 3 \end{aligned}$$

Both $f(-1)$ and $f(1)$ are equal to 3. This isn't enough to prove that $f(x)$ is even, because $f(x) = f(-x)$ must be true for all choices of x, not just a particular choice, in order for the function to be even. If we pick an arbitrary number a, then we get

$$\begin{aligned} f(-a) &= 3 \cdot (-a)^2 = 3 \cdot a^2 = 3a^2 \\ f(a) &= 3a^2 \end{aligned}$$

So we see that $f(-a) = f(a)$ for any choice of a. This shows that $f(x) = 3x^2$ is an even function.

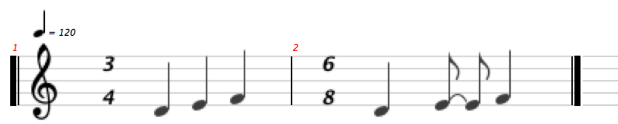
Chapter 9

Polyrhythms

Polyrhythms occur when two or more rhythms, which are perceived as conflicting, play simultaneously. An example is playing eighth notes with instrument (or limb for drummers or pianists) while another instrument plays eighth note triplets. This produces a 3 against 2 (or 2 against 3) polyrhythm. You can also get a 3 against 2 polyrythm by playing notes against their dotted counterparts, such as quarter notes against dotted quarter notes.

Some musicians might distinguish between 3 against 2 versus 2 against 3, but I don't. Distinguishing between them makes sense if one pulse is the dominant main pulse, but I also like to think of polyrhythms when both rhythms are equally in the foreground. Often this makes the meter ambiguous, meaning that you can interpret the music in more than one way. As an example consider the following two bars of music.

The first measure and the second measure sound exactly the same when played by a computer for both staves. A measure of 3/4 time has the same duration as a measure of 6/8 time, and both are equivalent to two dotted quarter notes. In the first measure, the first instrument playes three notes — D, E, F — while the second instrument plays two A in a row. This is a 3 against 2 polyrythm since we have one voice playing three notes while another plays two notes. The same is true in the second measure. The notation is slightly different because the second played note for the first instrument is presented as two tied eighth notes, but it sounds the same as a quarter note. For 6/8 time, the notation often makes it clear where the middle of the measure is: it falls exactly in the middle of the second played note in the first instrument.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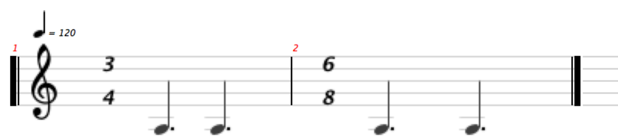
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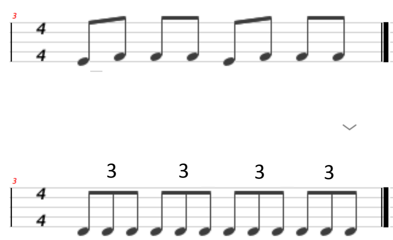

Figure 9.1: 3 against 2 (equivalently, 2 against 3) polyrhythm.

There is no single answer about whether the music presented here is *really* in 3/4 time or 6/8 time. You can think of it either way, it is a matter of convenience which way is notated, or one might be chosen because of the wider context of the music (waltzes are usually presented in 3/4 time, while many ballads are written in 6/8 time). If you have a measure with six equally spaced eighth notes and two accented notes (at the beginning and the middle), you would normally use 6/8 as the time signature. If the six equally spaced notes have three equally spaced accents, you might use 3/4 time instead. Often music might go back and forth between both types of accenting within the same piece of music, or even within the same melody, such as *America* from *West Side Story*.

Iin figure 9.2, the bottom voice plays 3 notes for every two in the top voice. By doubling the number of notes in the top figure, we get a 4 against 3 poyrhythm, with the top voice playing four notes for every three in the bottom voice.

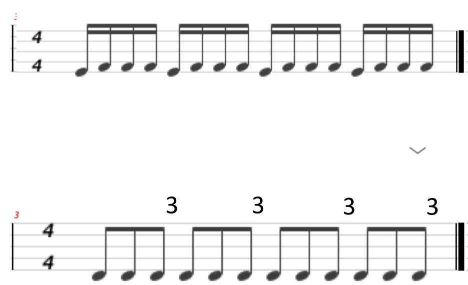

Figure 9.2: 4 against 3 polyrhythm.

Another way to generate polyrhythms is to accent notes at intervals that don't divide evenly into the number of notes per measure. For example, if there are 16 notes per measure, and we accent every third note, then the pattern of accents won't line up with the start of each measure. This can lead to playing "over the bar line" – a pattern that continues but ignores where the bar line is instead of starting over at the beginning of the next measure. An example is from the song "Bleed" by Meshuggah. The rhythmic figure that repeats in the lowest note (for the kick drum) repeats two thirty-second notes followed by two sixteenth notes over and over again. This process continues and doesn't stop when there is a new measure. So the second measure starts with two sixteenths followed by two thirty-second notes, and the third measure starts with one sixteenth followed by two thirty-second notes. On measure 4, the pattern starts repeating itself. Note that the highest note plays the beginning of each quarter note, regardless of what the kick is playing, which helps maintain a quarter note pulse. The second lowest note corresponds to the snare drum, which only plays on the third quarter note of each measure.

A useful tool for dealing with 4 against 3 polyrhythms is to write music in 12/8 time. This time signature is helpful because the number 12 is divisible by both 3 and 4, which makes it easier to notate some polyrhythms.

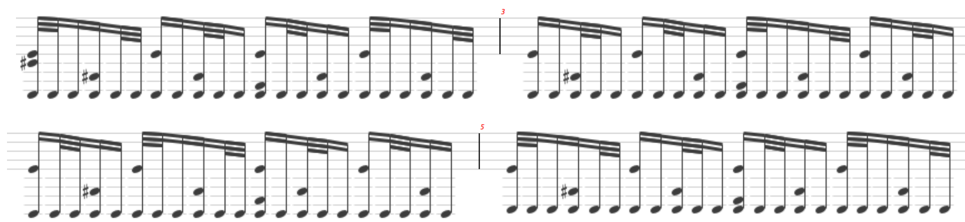

Figure 9.3: The first four measures of "Bleed" by Meshuggah. The first three measures are distinct (although highly similar). The fourth measure is identical to the first measure.

Chapter 10

Polyrhythms in music from Ghana, Cuba, and India

Chapter 11

Equal temperament versus just temperament tuning for scales

Pitches correspond to different numbers of vibrations per second, called Herz, and abbreviated Hz. A standard reference is to let the A above middle C on a piano be the note that vibrates at 440 Hz. Frequencies of other notes can be determined in relation to this standard. Historically, there hasn't always been a standard for A or any other note, and different orchestras have tuned to different values, but A = 440 started being used in the 1800s, although not universally. A guitar player might tune to one string on their guitar, not knowing whether this results in A=440 or not, so the actual frequency might be close but not exactly at 440 Hz.

There is a difficulty that there isn't a unique way to determine what frequencies other notes should be even if we agree that A = 440 Hz. A common way to determine intervals to other notes was discovered in ancient Greece over 2500 years ago. The octave occurs at notes that vibrate at twice the frequency, so a note vibrating at 880 Hz will sound an octave higher than A = 440 Hz, and a note vibrating at 220 Hz will sound an octave lower than A = 440 Hz. The perfect fifth above a note vibrates at roughly $3/2 = 1.5$ times the frequency of the reference note. We'll see that this is normally not exact with modern tuning.

Other intervals can also be expressed approximately as ratios

If these ratios were exact, then you could go up fifths from a starting note, and several octave laters, end up on a note that was the same (just higher pitched) as your starting note. Let's try this with fifths. We'll start with $A = 110$ (this is the second A below middle C).

Interval	ratio
octave	2 : 1
fifth (7 frets = 7 half-steps)	3 : 2
fourth (5 frets = 5 half-steps)	4 : 3
major third (4 frets = 4 half-steps)	5 : 4
minor third (3 frets = 3 half-steps)	6 : 5

approximate note	pitch
A	110
E	165
B	247.5
F#	371.25
C#	556.875
G#	835.3125
D#	1252.969
A#	1879.453
F	2819.18
C	4228.77
G	6343.154
D	9514.731
A	14272.1

Something has gone wrong, because if keep doubling A = 110, you'll always get integers. You'll also always get numbers that end in 0, but somehow we get a decimal of 14272.1 (which is rounded). If we keep doubling A=110 to see the frequencies of the octaves of A, we get Table 11.1. A standard 88-key piano ranges from A0= 27.5 Hz to A7=3520 Hz. So A9=14080 is quite high pitched, but we see it doesn't correspond very well to the frequency taken by adding fifths that always 1.5 times higher in frequency than the previous note. If A=14272.1, then we should have lower octaves of A would be $14272.1/2 = 7136.05$ and $14272.1/4 = 3568.024$ Hz. However, the highest A on a piano is 3520 Hz.

Sometimes the interval approach works well. Consider a minor triad consists of a root note, a note a minor third above that, and a note a major third above the second note. Thus, a major triad of the notes A3, C4, E4 (C4 being the middle C), should have a root note at A = 220 Hz. The frequency

Table 11.1: Frequencies for different octaves of A.

note	pitch
A2	110
A3	220
A4	440
A5	880
A6	1760
A7	3520
A8	7040
A9	14080

Table 11.2: Expected frequencies of notes minor thirds apart using the interval approach

approximate note	pitch (assuming ratios of 6/5)
A3	220
C4	$220 \times (6/5) = 264$
D#4	$220 \times (6/5)^2 = 316.8$
F#4	$220 \times (6/5)^3 = 380.16$
A4	$220 \times (6/5)^4 = 456.192$

of the second note, using the interval approach should 6/5 times the root frequency. This is $(220)(6/5) = 264$ Hz. The third note is a fifth above A3, or a minor third above C4, so using the interval approach, the frequency of the third note of the chord should be either

$$220 \times \frac{6}{5} \times \frac{5}{4} = 220 \times \frac{6}{4} = 220 \times 1.5 = 330$$

In this case, multiplying the intervals for the two thirds results in the expected interval for the fifth.

On the other hand, suppose we consider the notes A3, C4, D#4, F#4, A4. Each note is a minor third above the previous note. Using the interval approach, we'd expect the frequencies to be as listed in Table 11.2. But we'd like A4 to be 440 Hz, not 456.192 Hz.

The interval approach leads to problems if we want a system of notes where octaves are obtained by exactly doubling frequencies and the space between notes is exactly equal, so that, for example, the ratio of frequencies of two notes a minor third apart is the same, no matter which note you start with. In the example in Table 11.2, you could design an instrument with those frequencies except that you use A4 = 440. However a a chord played on A3 might sound like a different chord from one starting on D#4 because the ratios of the frequencies of the pitches will not be the same for the two chords.

Modern instruments tend to be designed so that melodies and chords sound the same when played in different keys, except for being higher or lower pitched, and this requires handling pitch frequencies a little differently.

Chapter 12

Probability

Here I'll just review the basics of probability. We think of random outcomes, such as the roll of a die, and describe probabilities in terms of events which are sets of outcomes. For example, A could be the event that I roll a 1, B could be the event that I roll a 2, and C could be the event that I roll and even number.

We think of probabilities as numbers between 0 and 1, which we write as $0 \leq P(A) \leq 1$, where $P(A)$ means the probability that event A occurs. Here are some properties of probabilities for any events A and B

- $0 \leq P(A) \leq 1$
- If $P(A) = 1$, then A is certain to occur
- If $P(A) = 0$, then A is impossible or certain to not occur
- $P(\text{not } A) = 1 - P(A)$
- If A and B cannot both happen, then A and B are called *mutually exclusive*. For example, if E is the event that I roll a number less than 4 and F is the event that I roll a 6, then E and F are mutually exclusive.
- If A and B are mutually exclusive, then $P(A \text{ and } B) = 0$.
- If A and B are mutually exclusive, then $P(A \text{ or } B) = P(A) + P(B)$.
- In general (whetehr or not A and B are mutually exclusive), $P(A \text{ or } B) = P(A) + P(B) - P(A \text{ and } B)$.

- If A and B are independent, then $P(A \text{ and } B) = P(A) \times P(B)$.
- If $0 < P(A) < 1$ and $0 < P(B) < 1$, then if A and B are mutually exclusive, then they are NOT independent.
- If $0 < P(A) < 1$ and $0 < P(B) < 1$, then if A and B are independent, then they are NOT mutually exclusive.
- $P(A|B)$ is read, "the probability of A given B". It means that the probability that A occurs when you know that B has occurred. For example, A might be the event that it rains in Albuquerque today, and B might be the event that it rains in Sante Fe today. Knowing B might change the probability that A occurs.
- $P(A|B) = \frac{P(A \text{ and } B)}{P(B)}$ is a formula for calculating conditional probabilities.
- If A and B are not independent, a useful formula is $P(A \text{ and } B) = P(A) \times P(B|A)$.
- If A and B are independent, then $P(A|B) = P(A)$ and $P(B|A) = P(B)$. This means that knowing B does not change the probability of A occurring, and knowing A does not change the probability of B occurring.
- Two ways to check for independence are: (1) check that $P(A \text{ and } B) = P(A) \times P(B)$, or (2) check that $P(A|B) = P(A)$ or $P(B|A) = P(B)$. You only have to check one of these. You don't need to check both conditions.