Practice Test 2

Formula Sheet

\[ \text{Var}(X) = E(X^2) - (E(X))^2, \quad \text{Cov}(X, Y) = E[XY] - E[X]E[Y] \]

1. Let \( X \) have pdf

\[
f_X(x) = \begin{cases} 
    x/2, & 0 < x < 2 \\
    0, & \text{otherwise}
\end{cases}
\]

(a) Find \( E[X] \)

\[
E[X] = \int_0^2 x^2/2 \, dx = \frac{x^3}{6} \bigg|_0^2 = \frac{4}{3}
\]

(b) Let \( Y = \sqrt{X} \). Find \( f_Y(y) \). Be sure to define \( f_Y(y) \) for \(-\infty < y < \infty\).

For \( 0 < y < \sqrt{2} \),

\[
F_y(y) = P(Y \leq y) \\
= P(\sqrt{X} \leq y) \\
= P(X \leq y^2) \\
= F_X(y^2)
\]

\[
\Rightarrow f_Y(y) = \frac{d}{dy} F_X(y^2) \\
= f_X(y^2) \cdot \frac{d}{dy} y^2 \\
= y^2/2 \cdot 2y \\
= y^3
\]

Thus,

\[
f_Y(y) = \begin{cases} 
    y^3, & 0 < y < \sqrt{2} \\
    0, & \text{otherwise}
\end{cases}
\]
(c) Find Cov(X, Y). Hint: Use the fact that Y = \sqrt{X} instead of trying to find the joint density for X and Y.

\[ \text{Cov}(X, Y) = \text{Cov}(X, \sqrt{X}) = E[X \cdot \sqrt{X}] - E[X]E[\sqrt{X}] \]

\[ E[X \sqrt{X}] = E[X^{3/2}] = \int_0^2 x^{5/2}/2 \, dx = (2/7)x^{7/2}/2 \bigg|_0^2 = (1/7)2^{7/2} = (8/7) \cdot \sqrt{2} \]

\[ E[\sqrt{X}] = \int_0^2 x^{3/2}/2 \, dx = (2/5)x^{5/2}/2 \bigg|_0^2 = (1/5)2^{5/2} = (4/5) \cdot \sqrt{2} \]

\[ \Rightarrow \text{Cov}(X, Y) = (8/7) \cdot \sqrt{2} - (4/5)(4/5) \cdot \sqrt{2} = \sqrt{2} \cdot ((8/7) - 16/15) = 0.1077496 \]

Note that as X increases, \( \sqrt{X} \) also increases, so it makes sense that the covariance is positive.

2. Let X and Y have joint density

\[ f_{X,Y}(x, y) = \begin{cases} k(2x + y), & x, y > 0 \\ 0, & \text{otherwise} \end{cases} \]

Solution. I needed to put an upper limit on x and y for this problem to make sense. I’ll use 0 < x, y < 1 but you could use whatever finite limits you want to have a slight variation on the problem.

(a) Find k

\[ k \int_0^1 \int_0^1 2x + y \, dx \, dy = 1 \]

\[ \Rightarrow k \int_0^1 x^2 + y x \bigg|_0^1 \, dy = 1 \]

\[ \Rightarrow k \int_0^1 1 + y \, dy = 1 \]

\[ \Rightarrow k(1 + 1/2) = 1 \]

\[ \Rightarrow k = 2/3 \]

(b) Find \( f_X(x) \)
for $0 < x < 1$,

$$f_X(x) = \int_0^1 f_{X,Y}(x,y) dy$$

$$= \left(\frac{2}{3}\right) \int_0^1 2x + y \ dy$$

$$= \left(\frac{2}{3}\right) \left(2xy + \frac{y^2}{2}\right) \bigg|_0^1$$

$$= \left(\frac{2}{3}\right)(2x + 1/2)$$

Thus,

$$f_X(x) = \left(\frac{2}{3}\right)(2x + 1/2) I(0 < x < 1)$$

(c) Find $f_Y(y)$

for $0 < y < 1$,

$$f_Y(x) = \int_0^1 f_{X,Y}(x,y) \ dx$$

$$= \left(\frac{2}{3}\right) \int_0^1 2x + y \ dx$$

$$= \left(\frac{2}{3}\right) \left(x^2 + xy\right) \bigg|_0^1$$

$$= \left(\frac{2}{3}\right)(1 + y)$$

Thus,

$$f_Y(y) = \left(\frac{2}{3}\right)(1 + y) I(0 < y < 1)$$

(d) Find Cov(X,Y)
\[ E[XY] = \frac{2}{3} \int_0^1 \int_0^1 xy(2x + y) \, dx \, dy \]
\[ = \frac{2}{3} \int_0^1 \int_0^1 2x^2y + xy^2 \, dx \, dy \]
\[ = \frac{2}{3} \int_0^1 2x^3y/3 + x^2y^2/2 \bigg|_0^1 \, dy \]
\[ = \frac{2}{3} \int_0^1 2y/3 + y^2/2 \, dy \]
\[ = \frac{2}{3} \int_0^1 2y^2/6 + y^3/6 \bigg|_0^1 \, dy \]
\[ = \frac{2}{3}(2/6 + 1/6) = 1/3 \]
\[ E[X] = \frac{2}{3} \int_0^1 2x^2 + x/2 \, dx \]
\[ = \frac{2}{3}\left(2x^2/3 + x^2/4\right) \bigg|_0^1 \]
\[ = \frac{2}{3}\left(2/3 + 1/4\right) = (2/3)(11/12) = 11/18 \]
\[ E[Y] = \frac{2}{3} \int_0^1 1 + y \, dx \]
\[ = \left(2/3\right)(y + y^2/2) \bigg|_0^1 \]
\[ = \left(2/3\right)(3/2) = 1 \]
\[ \text{Cov}(X, Y) = 1/3 - 11/18 = -5/18 \]

This suggests that as either \( X \) or \( Y \) go up, the other tends to go down. This is not obvious from the mathematical description. However, they do not appear to be independent because the joint density does not look like a product of a function of \( x \) and a function of \( y \).

(e) Find the density of the sum, \( f_{X+Y}(a) \).

Solution. This problem is a bit more difficult than I realized...
For this density, keep in mind that for \( Z = X + Y \), the support of \( Z \) is from 0 to 2. We can’t use the convolution formula for this because \( X \) and \( Y \) are not independent. The limits of integration are a bit tricky. We note that for \( 0 < z < 1 \), \( x \) ranges from 0 to \( z - y \) and \( y \) ranges from 0 to \( z \). If \( x > z - y \), then \( x + y > z \), which is not the range we are interested in, and if \( y > z \), then \( z - y < 0 \), so \( P(X \leq z - y) = 0 \), so again we are not interested in \( y > z \).
For $0 < z < 1$,

$$F_Z(z) = P(Z \leq z)$$
$$= P(X + Y \leq z)$$
$$= P(X \leq z - Y)$$

$$= \frac{2}{3} \int_0^1 \int_0^{1-z} f_{X,Y}(x, y) \, dx \, dy$$

$$= \frac{2}{3} \int_0^z \int_0^{z-y} 2x + y \, dx \, dy$$

$$= \frac{2}{3} \int_0^z x^2 + yx \bigg|_0^{z-y} \, dy$$

$$= \frac{2}{3} \int_0^z (z - y)^2 + y(z - y) \, dy$$

$$= \frac{2}{3} \int_0^z z^2 - 2yz + y^2 + yz - y^2 \, dy$$

$$= \frac{2}{3} \int_0^z z^2 - yz \, dy$$

$$= \frac{2}{3} \left( z^2 y - \frac{yz^3}{2} \right) \bigg|_0^z$$

$$= \frac{2}{3} \left( z^3 - z^3 / 2 \right)$$

$$= z^3 / 3$$

$$\Rightarrow f_Z(z) = z^2$$

For $1 < z < 2$, $x < z - y$ is automatically satisfied if $z - y > 1$, which occurs when $y < z - 1$, so we are interested in cases where $y > z - 1$. Consequently we integrate $y$ from $z - 1$ to 1.
For $1 < z < 2$,

\[ F_Z(z) = P(Z \leq z) = P(X + Y \leq z) = P(X \leq z - Y) = \int_{z-1}^{1} \int_{0}^{z-y} f_{X,Y}(x,y) \, dx \, dy \]

\[ = \left. \frac{2}{3} \left( z^2 y - y^2 \frac{z}{2} \right) \right|_{z-1}^{1} \]

\[ = \left. \frac{2}{3} \left( z^2 - z/2 - (z-1)^2 + z(z-1)^2/2 \right) \right|_{z-1}^{1} \]

\[ = \left. \frac{2}{3} \left( z^2 - z^2/2 - z^2/2 - z^2 + 2z - 1 + z^3/2 - z^2 + z/2 \right) \right|_{z-1}^{1} \]

\[ = (2/3)(z^3/2 - z^2 + 2z - 1) \]

\[ = z^3/3 - 2z^2/3 + 4z/3 - 2/3 \]

\[ \Rightarrow f_Z(z) = z^2 - 4z/3 + 4/3 \]

Thus,

\[ f_Z(z) = \begin{cases} 
  z^2 & 0 < z < 1 \\
  z^2 - 4z/3 + 4/3 & 1 \leq z < 2 \\
  0 & \text{otherwise}
\end{cases} \]
3. Let $X$ and $Y$ have joint mass function

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(a) Are $X$ and $Y$ independent? Justify your answer.

Solution. \( P(X = 1) = 0.35 \). \( P(Y = 1) = 0.2 \). \( P(X = 1, Y = 1) = 0.1 \neq P(X = 1)P(Y = 1) = 0.07 \). Thus, $X$ and $Y$ are not independent.

(b) Find the marginal distribution of $X$ by specifying the probability mass function for $X$.

\[
P(X = i) = \begin{cases} 
0.35 & i = 1 \\
0.25 & i = 2 \\
0.4 & i = 3 \\
0 & \text{otherwise}
\end{cases}
\]

The values are obtained for $P(X = 1)$ (as an example) by calculating

\[
P(X = 1) = P(X = 1, Y = 1) + P(X = 1, Y = 2) + P(X = 1, Y = 3)
\]