MATH 501

3-0235 — 50 SHEETS — 5 SQUARES
3-0236 — 100 SHEETS — 5 SQUARES
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COMET

Homework #6

16.6 Prove convergence using only the definition

(b) For any real number $k>0$, $\lim_{n\to\infty}\left(\frac{1}{n^k}\right)=0$

We need $\left|\frac{1}{n^k}-0\right|<\varepsilon$

$$\frac{1}{n^k}<\varepsilon \qquad (\text{since } n^k>0 \quad \forall n\in\mathbb{N}\ \forall k\in\mathbb{R}^+)$$

$$\frac{1}{\varepsilon}<n^k$$

$$\frac{1}{\sqrt[k]{\varepsilon}}<n \qquad (\text{since } k>0 \text{ the direction of the inequality doesn't change})$$

So, given $\varepsilon>0$, let $N=\frac{1}{\sqrt[k]{\varepsilon}}$. Then for any $n>N$ we have $\left|\frac{1}{n^k}-0\right|=\frac{1}{n^k}<\frac{1}{N^k}=\varepsilon \quad \forall k\in\mathbb{R}^+$

$$\Rightarrow \lim \frac{1}{n^k}=0 \quad \forall k\in\mathbb{R}^+$$

$\square$

(c) $\lim_{n\to\infty}\frac{3n+1}{n+2}=3$

We need: $\left|\frac{3n+1}{n+2}-3\right|<\varepsilon$

$$\left|\frac{-5}{n+2}\right|<\varepsilon$$

$$\frac{|-5|}{n+2}<\varepsilon \qquad (\text{since } n+2>0)$$

$$\frac{5}{\varepsilon}-2<n$$

So, given $\varepsilon>0$, let $N=\frac{5}{\varepsilon}-2$. Then for any $n>N$ we have $\left|\frac{3n+1}{n+2}-3\right|=\left|\frac{-5}{n+2}\right|=\frac{5}{n+2}<\frac{5}{\frac{5}{\varepsilon}-2+2}=\varepsilon$

$$\Rightarrow \lim_{n\to\infty}\frac{3n+1}{n+2}=3$$

$\square$

16.8 Show that the sequence is divergent.

(b) $b_n = (-1)^n$

Pf. Suppose b_n converges to the real number b.

Let $\varepsilon = 1$, by definition of convergence $\exists N$ s.t. if $n > N$

$\Rightarrow |b_n - b| < 1$. If n is odd and $n > N$

$\Rightarrow |-1-b| < 1$

$|(-1)(b+1)| < 1$

$|b+1| < 1 \quad \Rightarrow \quad -2 < b < 0$

If n is even and $n > N$

$\Rightarrow |1-b| < 1$

$|b-1| < 1 \quad \Rightarrow \quad 0 < b < 2$

b cannot satisfy both inequalities, we have a contradiction.

$\Rightarrow$ The sequence b_n is divergent.

$\square$

16.9 Prove or give a counterexample.

(a) If (s_n) converges to s, then $(|s_n|)$ converges to $|s|$.

Sol: In a previous homework we proved that

$$||x|-|y|| \leq |x-y| \checkmark$$

Now, consider $\quad ||s_n|-|s||$

applying the ineq. $\quad ||s_n|-|s|| \leq |s_n - s|$

but since $s_n \to s \Rightarrow \forall \varepsilon > 0 \ \exists N_1 \ni \forall n \in \mathbb{N},\ n > N_1 \Rightarrow |s_n - s| < \varepsilon$.

Now let $N = N_1$. Then for $n > N$, we have $n > N_1$

$$||s_n|-|s|| \leq |s_n - s| < \varepsilon$$

$$\Rightarrow ||s_n|-|s|| < \varepsilon$$

Thus $(|s_n|)$ converges to $|s|$.

$\square$

16.13 Suppose $(a_n), (b_n), (c_n)$ are sepuences s.t. $a_n \leq b_n \leq c_n$ $\forall n \in \mathbb{N}$ and s.t. $\lim a_n = \lim c_n = b$. Prove $\lim b_n = b$

Pf: Since $a_n \to b \Rightarrow \forall \varepsilon > 0 \; \exists \; N_1$ s.t $\forall n \in \mathbb{N}, n > N_1 \Rightarrow |a_n - b| < \varepsilon$

similarly, $c_n \to b \Rightarrow \forall \varepsilon > 0 \; \exists \; N_2$ s.t. $\forall n \in \mathbb{N}, n > N_2 \Rightarrow |c_n - b| < \varepsilon$

Let $N = \max\{N_1, N_2\}$

$\Rightarrow \forall \varepsilon > 0$, if $n > N \Rightarrow |a_n - b| < \varepsilon$ & $|c_n - b| < \varepsilon$ $\quad \forall n \in \mathbb{N}$

$\Rightarrow \quad -\varepsilon < a_n - b < \varepsilon \quad \& \quad -\varepsilon < c_n - b < \varepsilon$

$b - \varepsilon < a_n < b + \varepsilon \quad \& \quad b - \varepsilon < c_n < b + \varepsilon$

Since $a_n \leq b_n \leq c_n \Rightarrow \quad b - \varepsilon < a_n \leq b_n \leq c_n < b + \varepsilon$

$\Rightarrow \quad b - \varepsilon < b_n < b + \varepsilon$

$-\varepsilon < b_n - b < \varepsilon$

$|b_n - b| < \varepsilon$

In other words, $\lim_{n \to \infty} b_n = b$

$\square$

16.15 (a) Prove that x is an accumulation point of a set S iff there exists a sep. (s_n) of points in $S \setminus \{x\}$ s.t. (s_n) converges to x.

16.15(a) Prove x is an accumulation point of S iff $\exists$ a sequence (s_n) of points in $S \setminus \{x\}$ s.t. (s_n) converges to x.

a) ($\Rightarrow$) If x is an accumulation point $\forall \varepsilon > 0$ $N^*(x;\delta) \cap S \neq \emptyset$. Let $\delta = \frac{1}{n}$. Since $N^*(x;\frac{1}{n}) \neq \emptyset$, define s_n to be an element of $N^*(x;\frac{1}{n})$. Given any $\varepsilon > 0$, by the archimedian property there exists an integer N such that $\frac{1}{N} < \varepsilon$. For any $n > N$, $\frac{1}{n} < \frac{1}{N} < \varepsilon$.

For $n > N$, $|s_n - x| < \frac{1}{n} < \varepsilon$. Thus s_n converges to x.

($\Leftarrow$) suppose (s_n) converges to x, where $(s_n) \subseteq S \setminus \{x\}$. Since (s_n) converges to x, $\forall \varepsilon > 0$ $\exists N$ such that $|s_n - x| < \varepsilon$ for $n > N$. Now, this implies $s_n \in N^*(x;\varepsilon)$ since $s_n \in S \setminus \{x\}$, i.e. $N^*(x;\varepsilon) \cap S \neq \emptyset$.

16.15 (b) Prove that a set S is closed iff whenever (s_n) is a convergent sequence of points in $S \Rightarrow \lim s_n \in S$.

($\Leftarrow$) Whenever (s_n) is a conv. seq. of points in S it follows $\lim s_n \in S$ then S is closed.

We will prove the contrapositive.

Suppose S is not closed $\Rightarrow S' \nsubseteq S$ (the set of all accumulation points is not contained in S)

$\Rightarrow \exists x \in S' \ni x \notin S$.

Since $x \notin S \Rightarrow S \setminus \{x\} = S$

By part (a) and since x is an accumulation point, we know that there exists a sequence (s_n) of points in $S \setminus \{x\} = S$ s.t. (s_n) converges to x, i.e. $\lim s_n = x$.

In other words, there is a convergent sequence in S whose limit is not in S.

By taking the contrapositive, if all convergent seq. in S whose $\lim s_n \in S \Rightarrow S$ is closed.

($\Rightarrow$) Suppose S is closed and suppose (s_n) is a conv. seq. of points in S.

Also suppose $\lim s_n \notin S$.

$\Rightarrow s_n$ is a convergent sequence in $S \setminus \{x\}$.

By part (a), x has to be an accumulation point of S.

But S is closed, so $x \in S$, which contradicts the assumption that $x = \lim s_n \notin S$.

$\Rightarrow$ If S is closed, whenever (s_n) is a conv. seq. of points in $S \Rightarrow \lim s_n \in S$.

$\square$

17.5 Determine convergence and find limits that exist

(a) $S_n = \dfrac{2^{3n}}{3^{2n}}$

Using the ratio test, consider $\dfrac{S_{n+1}}{S_n} = \dfrac{\frac{2^{3(n+1)}}{3^{2(n+1)}}}{\frac{2^{3n}}{3^{2n}}} = \dfrac{2^3}{3^2}$

$$= \frac{8}{9} = L < 1$$

Since $\frac{S_{n+1}}{S_n} \to L$, and $L<1 \Rightarrow \lim S_n = 0$.

S_n converges.

(f) $S_n = \dfrac{3+n-n^2}{1+2n}$

Finding a lower bound for the numerator:

$$3+n-n^2 \leq n^2 \qquad \text{for} \quad n>1$$

(see proof on next page)

An upper bound for the denominator:

$$1+2n \geq -2n \qquad \forall n \in \mathbb{N}$$

$$\Rightarrow S_n = \frac{3+n-n^2}{1+2n} \leq \frac{n^2}{-2n} = -\frac{n}{2}$$

To make this smaller than any M we want $n > -2M$.
We need $n>1$ and $n>-2M$.

Given any $M \in \mathbb{R}$, let $N = \max\{1, -2M\}$. Then $n > N \Rightarrow$ $n>1$ and $n>-2M$. Since $n>1$ we have $3+n-n^2 \leq n^2$ and $1+2n \geq -2n$. So, for $n>N$ we have

$$\frac{3+n-n^2}{1+2n} \leq \frac{n^2}{-2n} = -\frac{n}{2} < M$$

$$\Rightarrow \lim \frac{3+n-n^2}{1+2n} = -\infty$$

The sequence diverges.

CONT...

17.5 (f) CONT...

Proof that $3+n-n^2 \leq n^2$ if $n>1$

By induction. If $n=2$ $3+2-4 = 1 \leq 2^2 = 4$ ✓

Suppose $3+n-n^2 \leq n^2$ is true for $n=k$

$\Rightarrow 3+k-k^2 \leq k^2$

Now,

$$\begin{aligned} 3+(k+1)-(k+1)^2 &= 3+k+\cancel{1}-k^2-2k-\cancel{1} \\ &= (3+k-k^2)-2k \\ &\leq k^2-2k \quad \text{(by induction hyp.)} \\ &\leq k^2 \quad \text{(since } k>1) \\ &\leq k^2+2k+1 \\ &= (k+1)^2 \end{aligned}$$

$\Rightarrow$ If $n>1$ then $3+n-n^2 \leq n^2$ $\square$

17.6 Prove or give counterexample.

(a) If (s_n) and (t_n) are divergent sequences, then (s_n+t_n) diverges

FALSE. Consider $s_n = (-1)^n$ and $t_n = -(-1)^n$

s_n diverges because it alternates between -1 and 1
t_n diverges for the same reasons.

But $s_n+t_n = (-1)^n - (-1)^n = 0$ is convergent to 0.

$\square$

(b) If (s_n) and (t_n) are divergent sequences $(s_n t_n)$ diverges.

FALSE Consider the same sequences as above:

$s_n = (-1)^n$ and $t_n = -(-1)^n$

Both of them are divergent However:

$(s_n t_n) = -(-1)^n(-1)^n = -(-1)^{2n}$

$= \{-1,-1,-1,\dots\}$ is a sequence that converges to -1.

17.7 Give an example of an unbounded seq. that does not diverge to $+\infty$ or to $-\infty$.

Consider $s_n = (-1)^n n$

The sequence is unbounded because we cannot find an $M \geq 0$ s.t. $|s_n| \leq M$.

It is not diverging to ∞ because we cannot find an $M_1 \in \mathbb{R}$ s.t. if $n > N \Rightarrow s_n > M_1$

It is also not diverging to $-\infty$ because we cannot find an $M_2 \in \mathbb{R}$ s.t. if $n > N \Rightarrow s_n < M_2$.

17.8 (a) Example of convergent seq. of positive numbers s.t. $\lim\left(\frac{s_{n+1}}{s_n}\right) = 1$.

Consider the sequence $s_n = 1$. $\forall n$

Clearly $\lim_{n\to\infty}\left(\frac{s_{n+1}}{s_n}\right) = \lim_{n\to\infty} \frac{1}{1} = 1$

Also the sequence is constant $\forall n$, it converges to 1.

(b) Example of divergent sequence of positive numbers s.t. $\lim\left(\frac{t_{n+1}}{t_n}\right) = 1$

Consider $t_n = n$

$\lim_{n\to\infty}\left(\frac{t_{n+1}}{t_n}\right) = \lim_{n\to\infty} \frac{n+1}{n} = \lim_{n\to\infty} 1 + \frac{1}{n} = 1$

However $t_n = n$ is divergent.

17.18 Suppose (s_n) is a convergent seq. with $a \leq s_n \leq b$ $\forall n \in \mathbb{N}$. Prove $a \leq \lim s_n \leq b$

Proof: Consider the set $S = [a, b]$. This set is closed and we have a sequence s_n of points in S.

CONT...

17.18 CONT...

By problem 16.15(b). We know that $\lim_{n\to\infty} s_n \in S$

i.e. $\lim_{n\to\infty} s_n \in [a,b]$ i.e. $a \leq s_n \leq b$.

$\square$

18.3 Prove that each seq. is monotone & bounded and find the limit

(b) $s_1 = 2$ $\quad s_{n+1} = \frac{1}{4}(s_n + 5)$ $\quad n \in \mathbb{N}$

* Proof of monotony by induction (the sequence is decreasing)

- Show: $s_1 > s_2$: $\quad s_1 = 2 > \frac{1}{4}(2+5) = 1.75 = s_2$
- Suppose that for $k \in \mathbb{N}$ $\quad s_k > s_{k+1}$
- Then $\quad s_{k+1} = \frac{1}{4}(s_k + 5) > \frac{1}{4}(s_{k+1} + 5) = s_{k+2}$

The induction step holds, so $\underline{s_n > s_{n+1}}$ $\quad \forall n \in \mathbb{N}$

* Proof that s_n is bounded below 1.

- Show: $s_1 = 2 > 1$
- Suppose that for $k \in \mathbb{N}$: $\quad s_k > 1$
- Then $\quad s_{k+1} = \frac{1}{4}(s_k + 5) = \frac{1}{4}s_k + \frac{5}{4} > \frac{1}{4} + \frac{5}{4} > 1$

$\Rightarrow$ $\underline{s_n > 1}$ $\quad \forall n \in \mathbb{N}$ (s_n is bounded below by 1)

Since s_n is a monotone decreasing sequence bounded by the interval $[1,2]$ $\Rightarrow$ s_n is convergent

* Find the limit

Since $\lim s_{n+1} = \lim s_n$ (see 16.11) and letting $\lim s_n = s$

$$s = \frac{1}{4}(s+5) \Rightarrow 4s = s+5 \Rightarrow 3s = 5 \Rightarrow s = \frac{5}{3}$$

$$\Rightarrow \underline{\underline{\lim_{n\to\infty} s_n = \frac{5}{3}}}$$

18.4 Find examples of sequences of real numbers s.t.

(a) Cauchy but not monotone

Consider $s_n = \frac{(-1)^n}{n}$

It is not monotone. Consider the first 3 terms:

$s_1 = -1 \qquad s_2 = \frac{1}{2} \qquad s_3 = -\frac{1}{3}$

Clearly $s_1 < s_2$ but $s_2 > s_3$.

- It is Cauchy because it converges to 0, and we know a sequence of real numbers is convergent iff it is Cauchy.

(b) Monotone, but not Cauchy

Consider $s_n = n$

- It is monotone. Evidently $s_{n+1} > s_n$ always because $n+1 > n$ for $n \in \mathbb{N}$
- It is not Cauchy: $|s_m - s_n| \geq 1$ for any m, n $m \neq n$.

(c) Bounded, but not Cauchy.

Consider $s_n = (-1)^n$

- It is bounded because $|s_n| \leq 1. \quad \forall n \in \mathbb{N}$
- It is not Cauchy because for any 2 adjacent terms n and $n+1$

$$|s_n - s_{n+1}| = 2 \qquad \forall n \in \mathbb{N}$$

so, we don't have that $\forall \varepsilon > 0 \; \exists N$ st $m, n > N \Rightarrow |s_n - s_m| < \varepsilon$

18.14 $s_n = (1 + \frac{1}{n})^n$. Use binomial theo. to show that (s_n) is increasing with $s_n < 3 \; \forall n$. Conclude that (s_n) is convergent.

Sol: - Show s_n is increasing.

Applying the binomial theorem to s_n we have ...

CONT...

18.14 CONT...

$$s_n = \left(1+\frac{1}{n}\right)^n = \sum_{i=0}^{n}\binom{n}{i}\frac{1}{n^i}$$

By induction (using the binomial theorem) we will prove that s_n is increasing.

* $s_1 = 2 < s_2 = 2.25$ ✓

* $s_{k+1} = \left(1+\frac{1}{k+1}\right)^{k+1} = \sum_{i=0}^{k+1}\binom{k+1}{i}\frac{1}{(k+1)^i} > s_n = \sum_{i=0}^{k}\binom{k}{i}\frac{1}{k^i}$

* $s_{k+2} = \left(1+\frac{1}{k+2}\right)^{k+2} = \sum_{i=0}^{k+2}\binom{k+2}{i}\frac{1}{(k+2)^i} = \sum_{i=0}^{k+2}\binom{k+1}{i}\frac{1}{(k+2)^i} + \sum_{i=0}^{k+2}\binom{k+1}{i-1}\frac{1}{(k+2)^i}$

$$> \sum_{i=0}^{k+1}\binom{k+1}{i}\frac{1}{(k+1+1)^i} + \sum_{i=0}^{k+2}\binom{k+1}{i-1}\frac{1}{(k+1+1)^i}$$

$$> \sum_{i=0}^{k+1}\binom{k+1}{i}\frac{1}{(k+1)^i} = s_{k+1}$$

Similarly, we will prove that $s_n < 3$

* $s_1 = 2 < 3$ ✓

* Suppose true for $k \in \mathbb{N}$ $\Rightarrow$ $s_k = \sum_{i=0}^{k}\binom{k}{i}\frac{1}{k^i} > 3$

* $s_{k+1} = \sum_{i=0}^{k+1}\binom{k+1}{i}\frac{1}{(k+1)^i}$

$$> \sum_{i=0}^{k}\binom{k+1}{i}\frac{1}{(k+1)^i} > \sum_{i=0}^{k}\binom{k+1}{i}\frac{1}{\sum_{j=0}^{i}\binom{i}{j}k^j}$$

$$> 3$$

$\Rightarrow$ Since s_n is increasing and bounded $\Rightarrow$ it converges

$\square$