23.2) \( f(x) = \frac{e^x}{x} \) on \([2,5] \) is uni. cont.

\( e^x \) is cont. for \( x \in \mathbb{R} \) \( \frac{1}{x} \) is cont. on \( x \in \mathbb{R} \setminus \{0\} \)

so \( \frac{e^x}{x} \) is cont. on \( x \in \mathbb{R} \setminus \{0\} \) \([2,5] \subseteq \mathbb{R} \setminus \{0\} \) and \([2,5] \) is a compact domain so \( \frac{e^x}{x} \) is uni. cont.

e) \( f(x) = \frac{1}{x^2} \) on \((0,1) \) not uni. cont. \( f(x) \) is uni. cont.

@n \((a, b)\) if f is cont. on \([a, b] \) but \( \frac{1}{x^2} \) is not cont at \( x = 0 \).

23.5) \( f(x) = \sqrt{x} \) is uni. cont. on \([0, \infty) \)

\( f \) is cont. on \([0, 2] \) is compact so \( f \) is uni. cont. on \([0, 2] \)

for \((1, \infty) \) \( |\sqrt{x} - \sqrt{y}| < \varepsilon \) when \( |x - y| < \delta \) and \( \sqrt{x}, \sqrt{y} > 1 \) so

\[ \frac{|x - y|}{1 - \sqrt{x} - \sqrt{y}} = \frac{|x - y|}{1 - \sqrt{x} + \sqrt{y}} \leq |x - y| \leq \delta \]

let \( \varepsilon = \delta \) then \( |x - y| < \delta \) \( |\sqrt{x} - \sqrt{y}| < \varepsilon \)

so \( f(x) \) is uni. cont. on \((1, \infty) \)

let \( \delta_1 \) be the delta for \([0,2] \) and \( \delta_2 \) for \((1, \infty) \) then

\[ \delta_2 = \min(\delta_1, \delta_2) \]

if \( |x - y| < \delta_3 \) we either have

1) \( x, y \in [0, 2] \)
and \( |x - y| < \delta_3 \leq \delta_1 \) \( \Rightarrow |\sqrt{x} - \sqrt{y}| < \varepsilon \)

2) \( x, y \in (1, \infty) \)
and \( |x - y| < \delta_3 \leq \delta_2 \) \( \Rightarrow |\sqrt{x} - \sqrt{y}| < \varepsilon \).
3) \(x \in [0,2]\ \&\ y \in (1,\infty)\)

\[\Rightarrow x \in [1,2] \Rightarrow x, y \in (1,\infty)\]

\[\text{by case 2}\]

\[|\sqrt{x} - \sqrt{y}| < \varepsilon\]

\[\text{or } y \in (x, x+1) \leq [0,2]\]

\[\text{by case 1}\]

\[|\sqrt{x} - \sqrt{y}| < \varepsilon\]

\(\text{and } \text{fix} \text{ is uniformly continuous.}\)

23.15 since \(\text{fix} = f(x+2) \forall x \in \mathbb{R}\) restrict \(\text{fix} = f(x)\) to \([0,2]\).

\(\text{fix} = f(x)\) is continuous on \([0,2]\), which is compact and

\(\text{hence } \text{fix} = f(x)\) is bounded on \([0,2] \). Now since \(\text{fix} = f(x)\)

is bounded on \([0,2] \), \(f_x \in \mathbb{R}, \text{ for } x \in \mathbb{Z}\) with \(x + n \in [0,2] \).

Thus \(|f(x+n\varepsilon)| = |f(x)| < M\), the bound for \(\text{fix} = f(x)\) on \([0,2]\).

Again since \(\text{fix} = f(x)\) is continuous on \([0,2]\), \(\text{fix} = f(x)\) is

uniformly continuous on \([0,2]\). In fact, \(\text{fix} = f(x)\)

is uniformly continuous on \([(n-1)\varepsilon, n\varepsilon]\) for \(n \in \mathbb{Z}\) with

the same \(\delta\), i.e., if \(|x-y| < \delta \Rightarrow f(x) - f(y) < \varepsilon/2\)

\(\text{note if } x, y \in [(n-1)\varepsilon, n\varepsilon] \Rightarrow |x-y| < \delta \Rightarrow |x-(n-1)\varepsilon - (y-n\varepsilon)| < \delta\)

but \(|f(x-(n-1)\varepsilon) - f(y-(n-1)\varepsilon)| = |f(x) - f(y)| < \varepsilon/2\)

Note, if \(|x-y| < \delta \), then either \((n-1)\varepsilon < x, y < n\varepsilon\) some \(n \in \mathbb{Z}\)

or \(x > n\varepsilon < y\) some \(n\) (or \(y > n\varepsilon < x\) some \(n\))

\(|x-y| \leq |x-n\varepsilon| + |n\varepsilon-y| < \delta\) \(\Rightarrow |f(x) - f(y)| < \varepsilon/2\)

\[\text{and } \text{fix} = f(x)\)

is uniformly continuous for all reals.\]
23. (b) \( \forall \varepsilon > 0 \) \( \exists \delta > 0 \) : \( |x - 5| < \delta \) implies \( |15(x) - 5(5)| < \varepsilon \)

\( 0 < \delta < \frac{\varepsilon}{3} \) \( \therefore \) \( |x - 5| < \frac{\varepsilon}{3} \) implies \( |15(x) - 5(5)| < \varepsilon \)

We need to show \( |x - 5| < \delta \) implies \( |15(x) - 5(5)| < \varepsilon \)

\begin{align*}
2\;|x - 5| &< \delta_1 + \delta_2 \implies 15\;|x - 5| < 15\;\delta_1 + 15\;\delta_2 \\
&< 3\varepsilon + 3\varepsilon \quad \text{let} \quad \delta_3 = \frac{\delta_1 + \delta_2}{2} \\
\therefore \quad |x - 5| &< \delta_3 \implies |15(x) - 5(5)| < 3\varepsilon + 3\varepsilon \\
&= 3\varepsilon = \frac{3}{\alpha} = \frac{3}{1} = 3
\end{align*}

23.(10) \( f(x) = g(x) = x \) \( \forall x\in\mathbb{R} \), \( g(x) \) are uniformly cont. \( \forall x\in\mathbb{R} \)

But \( f(x) = x^2 \) is not

see attached for 23.15

25. (a) \[ f(x) = x^2 \sin\left(\frac{1}{x}\right) \quad \text{as} \quad x \to 0 \quad \text{and} \quad f(0) = 0 \]

\[ f'(x) = 2x \sin\left(\frac{1}{x}\right) - \cos\left(\frac{1}{x}\right) \quad \text{except at} \quad x = 0 \]

\[ \lim_{x \to 0} f'(x) = \frac{\lim_{x \to 0} f(x) - f(0)}{x - 0} = \lim_{x \to 0} \frac{f(x) - f(0)}{x} \]

\( \lim_{x \to 0} x \sin\left(\frac{1}{x}\right) = 0 \)

Since \( \lim_{x \to 0} x \sin\left(\frac{1}{x}\right) \leq \lim_{x \to 0} |x| = 0 \)

\[ 0 \leq f'(0) = 0 \]
\[ 25.6 \quad (c) \quad \lim_{x \to 0} 2x \sin(1/x) - \cos(1/x) \]

notice that the first part goes to 0 since

\[ \lim_{x \to 0} 2x \sin(1/x) \leq \lim_{x \to 0} (2x) = 0 \]

but the second part oscillates between -1 and 1 so

the limit cannot exist. So \( f' \) is not cont at \( x=0 \)

\[ 25.8 \quad f(x) = x^2 \sin(1/x^2) \text{ for } x \neq 0 \quad f(0) = 0 \]

a) \[ f'(x) = 2x \sin(1/x^2) - 2 \cos(1/x^2) \]

which is well defined except for possibly at \( x=0 \) so \( f \) is diff on \( \mathbb{R}/\{0\} \)

for \( x=0 \)

\[ \lim_{x \to 0} \frac{f(x) - f(0)}{x - 0} = \lim_{x \to 0} x \sin(1/x^2) \leq \lim_{x \to 0} x = 0 \]

\[ = -\lim_{x \to 0} x = 0 \]

so \( f \) is also diff at \( x=0 \) and \( f'(0) = 0 \)

b) \[ \lim_{x \to 0} f'(x) = \lim_{x \to 0} 2x \sin(1/x^2) - 2 \cos(1/x^2) = \lim_{x \to 0} \cos(1/x^2) \]

the \( \cos(1/x) \) will keep oscillating between 1 and -1 while

the denominator gets smaller blowing it up so limit
do not exist.

\[ 25.11 \quad f(x) = x^2 \text{ if } x \in \mathbb{Q} \text{ and } f(x) = 0 \text{ if } x \text{ is irrational} \]

a) let \( x \neq 0 \) be a rational then there exists a sequence

of irrationals approaching \( x \) since rationals are dense

in reals. But the value at all the sequence points is 0

so if we make \( \varepsilon \) less than the value at \( x \) we can

find a \( \delta \) arbitrarily small with would make \( f(x) - f(x+\delta) > \varepsilon \)

since we can make \( f(x+\delta) \) be an irrational \( f(x) - 0 > \varepsilon \)
Suppose \( f \) differentiable at \( x = 0 \)

If \( x \) rational, then there exist a sequence \( x_n \) of irrational numbers with

\[
\lim_{n \to \infty} x_n = x,
\]

and \( f \) diff at \( x \) if

\[
\lim_{n \to \infty} \frac{f(x)-f(x_n)}{x-x_n} = \lim_{n \to \infty} \frac{x^2}{x-x_n}
\]

which does not exist as the numerator is fixed and the denominator becomes increasingly small.

If \( x \) irrational, there exists a sequence \( x_n \) of rationals with \( \lim_{n \to \infty} x_n = x \), \( f \) diff at \( x \) if

\[
\lim_{n \to \infty} \frac{f(x)-f(x_n)}{x-x_n} = \lim_{n \to \infty} \frac{x^2}{x-x_n}
\]

which also doesn't exist.

However, if \( x = 0 \)

\[
\lim_{x \to 0} \frac{f(x)-f(0)}{x} = \lim_{x \to 0} \frac{f(x)}{x}
\]

\[
\lim_{x \to 0} \frac{f(x)}{x} = \lim_{x \to 0} \frac{x^2}{x} = 0
\]

so \( f \) is diff at \( x = 0 \)

\[
\lim_{x \to 0} \frac{0}{x} = 0
\]
25.12) \((x-a)^2 \mid P(x)\) then \(P(x) = Q(x)\) where \(Q(x)\) is a polynomial

\[ P(x) = (x-a)^2 Q(x) \Rightarrow P'(x) = 2(x-a) Q(x) + (x-a)^2 Q'(x) \]

Polynomials are closed under differentiation, multiplication, and addition so \(2Q(x) + (x-a)Q'(x)\) is also a polynomial

\[ (x-a) \left[ 2Q(x) + (x-a)Q'(x) \right] = (x-a)^2 Q(x) \cdot G(x) = P'(x) \]

and \((x-a) \mid (x-a) \cdot G(x)\) is true so \((x-c) \mid P'(x)\) is true

26.6) \(f(x) = x\) is cont. and diff. but \(f'(x) = 1 \neq 0\)

\[ f(x) = x \text{ is cont and } f(-a) = f(a) \text{ but} \]

\[ f'(x) = 1 \text{ for } x > 1 \]

\[ f'(x) = -1 \text{ for } x < 1 \]

\[ f(x) = x^3 \text{ on } [1, 2] \text{ f is cont and diff but} \]

\[ f'(x) = 3x^2 \quad 3 \leq f'(x) \leq 12 \quad \text{so} \quad f'(x) \neq 0 \text{ for } x \in [1, 2] \]

26.9) a) if \(f'(x) > 0\) \(\forall x \in I\) then \(f\) is strictly increasing

b) \(f'(x) < 0\) \(\ldots \) \(f\) is strictly decreasing

Converse: if \(f\) is strictly increasing then \(f'(x) > 0\) \(\forall x \in I\)

a) Let \(f(x) = x^3\) then \(\forall x, y \in \mathbb{R} : x < y \implies x^3 < y^3\)

\[ \text{but } f'(x) \neq 0 \text{ since } f'(0) = 0 \]

b) Let \(f(x) = -x^3\) \(\ldots \) \(\implies x^3 > y^3\)

\[ \text{but } f'(x) \neq 0 \text{ since } f'(0) = 0 \]
26.11) \( f \) is diff on \([a, b]\), \( f'(x) \geq 0 \quad \forall x \in [a, b] \)

and \( f'(a) = 0 \quad \forall A \leq [a, b] \) show \( f \) is strictly inc.

From problem 26.8 we know that \( f \) is increasing.

Also since \( f'(a) = 0 \) can only be true if \( A \) has measure 0

so \( f(x) < f(y) \) if \( x < y \). Since if \( f(x) = f(y) \) and

\( x < y \) then by IVT either \( f'(x) = 0 \) \( \forall x < z < y \)

which was not allowed or \( f'(z) < 0 \) which is also not

allowed by definition of problem.

26.13) a) \( f(0) = 0 \), \( f(1) = 1 \), \( f(2) = 2 \) \( f \) is diff on \( \mathbb{R} \)

show \( \exists \sigma_1 \in (0, 1) : f'(c_1) = 2 \) automatic from IVT

show \( \exists \sigma_2 \in (1, 2) : f'(c_2) = 0 \) automatic from Rolle's Theom

b) show \( \exists \sigma_2 \in (1, 2) : f'(c_2) = 0 \) automatic from IVT for Derivatives

\( c) \) show \( \exists \sigma_3 \in (0, 2) : f'(c_3) = \frac{5}{4} \) automatic from IVT for Derivatives

and \( 0 \leq \sigma < 2 \)

26.23d) \( |f(x) - f(y)| \leq M|x - y|^{a} \quad \text{if} \quad |x - y| < \delta \)

\[ \text{let} \quad \delta^{a} = \delta^{a} \]

Then \( |x - y|^{a} < \delta^{a} \) implies \( |f(x) - f(y)| < \delta \quad \text{so} \quad |x - y| < \delta \)

\( f \) is uni cont.

b) \( a = 1 \) then \( f \) is not necessarily diff.

c) \( a = 1 \) then \( f \) is not necessarily diff.

\( f = 1 \times 1 \) is Lipschitz since \( |1x - 1y| \leq M|x - y| \)

\( f \) is not diff at \( x = 0 \)

d)
6) \( \lim_{x \to c} \frac{f(x) - g(x)}{x - c} \leq \lim_{x \to c} M \frac{|x - c|}{x - c} = M \lim_{x \to c} (x - c) = 0 \) for all \( c \). So \( f(x) = g(x) \) for all \( x \). Since \( \lim_{x \to c} f(x) \) was true, \( \lim_{x \to c} g(x) \) must also be true.