Homework #9

$\boxed{26.6}$ Three functions:

1) $\sin(x)$ on $[0,1]$. This function is continuous and differentiable on $[0,1]$, but $\sin(0) \neq \sin(1)$. There is no pt $c \in (0,1)$ s.t. $\frac{d}{dx}\sin(x) = 0$.

2) $|x|$ on $[-5,5]$. This function is continuous and $|-5| = |5|$, but $|x|$ is not differentiable at $x = 0$. So, there is no pt. $c \in (-5,5)$ s.t. $\frac{d}{dx}|x| = 0$.

3) $\frac{1}{|x|}$ on $[-1,0) \cup (0,1]$. This function is differentiable and $\frac{1}{|-1|} = \frac{1}{|1|}$, but is not continuous. There is no $c \in (-1,0) \cup (0,1)$ s.t. $\frac{d}{dx}\frac{1}{|x|} = 0$.

$\boxed{26.8}$ (a) f is increasing on I iff $f'(x) \geq 0 \ \forall x \in I$.

Proof. Suppose f is increasing on I. Then $x_i < x_j$ on $I \Rightarrow f(x_i) \leq f(x_j) \ \forall i<j$.

Since f is differentiable, $f'(x_j) = \lim_{x_i \to x_j} \frac{f(x_i) - f(x_j)}{x_i - x_j}$.

But $f(x_i) \leq f(x_j) \ \forall x_i < x_j$, so $f'(x_j) \geq 0 \ \forall j > i$.

$f'(x_i) = \lim_{x_j \to x_i} \frac{f(x_j) - f(x_i)}{x_j - x_i} \geq 0 \ \forall i<j$ since $x_j > x_i$.

$\Rightarrow f'(x) \geq 0 \ \forall x \in I$.

Now suppose $f'(x) \geq 0 \ \forall x \in I$.

By MVT, $\Rightarrow \exists c \in I$ s.t. $f(x_j) - f(x_i) = f'(c)(x_j - x_i)$, $x_i < x_j$.

If $f'(c) = 0$, $f(x_i) = f(x_j)$.

If $f'(c) > 0$, then by Thm 26.8 $f(x_j) > f(x_i)$.

$\Rightarrow f(x_j) \geq f(x_i) \ \forall x_i < x_j \in I$, so f is increasing on I. ∎

(b) f is decreasing on I iff $f'(x) \leq 0 \ \forall x \in I$.

Proof. f decreasing on $I \Rightarrow$ for $x_i < x_j$, $f(x_i) \geq f(x_j) \ \forall i<j$.

Since f is differentiable, $f'(x_j) = \lim_{x_i \to x_j} \frac{f(x_i) - f(x_j)}{x_i - x_j}$.

But $f(x_i) \geq f(x_j) \ \forall x_i < x_j$, so $f'(x_j) \leq 0 \ \forall j > i$.

$f'(x_i) = \lim_{x_j \to x_i} \frac{f(x_j) - f(x_i)}{x_j - x_i} \geq \ \forall i<j$ since $x_j > x_i$.

$\Rightarrow f'(x) \leq 0 \ \forall x \in I$.

Now suppose $f'(x) \leq 0 \ \forall x \in I$.

By MVT, $\Rightarrow \exists c \in I$ s.t. $f(x_j) - f(x_i) = f'(c)(x_j - x_i)$, $x_i < x_j$.

If $f'(c) = 0$, $f(x_i) = f(x_j)$

If $f'(c) < 0$, then Thm 26.8 $\Rightarrow f(x_i) < f(x_j)$.

$\Rightarrow f(x_j) \geq f(x_i) \quad \forall\ x_i < x_j,\ i<j$, so f is decreasing. ■

$\boxed{26.18}$ f is diff. on (a,b) & cont. on $[a,b]$, with $f(a)=f(b)=0$. Prove that $\forall k \in \mathbb{R}\ \exists c \in (a,b)$ s.t. $f'(c) = kf(c)$, where $g(x) = e^{-kx}f(x)$.

Proof. Since f is continuous on $[a,b]$ and differentiable on (a,b), and e^{-kx} is cont. & diff. as well for any $k\in\mathbb{R}$, then $g(x)=e^{-kx}f(x)$ is continuous on $[a,b]$ and diff. on (a,b).

Because $f(a)=f(b)=0$, $g(a) = e^{-ka}f(a) = 0 = e^{-kb}f(b) = g(b)$.

By Rolle's Thm $\Rightarrow \exists c \in (a,b)$ s.t. $g'(c)=0$.

Since $g'(x) = -ke^{-kx}f(x) + e^{-kx}f'(x)$ (product rule for derivatives),

then $g'(c) = 0 = -ke^{-kc}f(c) + e^{-kc}f'(c)$.

$\Rightarrow e^{-kc}f'(c) = ke^{-kc}f(c)$

Cancelling out the e^{-kc} term on both sides yields

$f'(c) = kf(c)$.

This holds $\forall k \in \mathbb{R}$ since k was chosen arbitrarily. ■

$\boxed{26.23}$ f is defined on I, and suppose $\exists\ M, \alpha > 0$ s.t. $|f(x)-f(y)| \leq M|x-y|^{\alpha}\ \forall x,y \in I$.

ⓐ f is uniformly continuous on I.

Proof. f is unif. cont. means $\forall \epsilon>0\ \exists \delta>0$ s.t. $|f(x)-f(y)|<\epsilon$ wherever $|x-y|<\delta$, $x,y\in I$. Let $\delta = (\epsilon/M)^{1/\alpha}$ since $M>0$.

Because $\delta>0$, and if $|x-y|<\delta$, $|x-y|^{\alpha} < \epsilon/M$, $\alpha>0$.

Then we have that $|f(x)-f(y)| \leq M|x-y|^{\alpha} < M\frac{\epsilon}{M} = \epsilon$.

So f is uniformly continuous. ■

ⓑ $\alpha > 1 \Rightarrow f$ is constant.

Proof. f is differentiable:

Def. of differentiable: $\lim_{x\to c} \frac{f(x)-f(c)}{x-c}$ exists.

Setting $c=y$, we have that $\lim_{x\to y}\frac{f(x)-f(y)}{x-y}$, $x,y\in I$.

But $\lim_{x\to y}\frac{f(x)-f(y)}{x-y} \leq \lim_{x\to y}\frac{|f(x)-f(y)|}{|x-y|} \leq \lim_{x\to y} M|x-y|^{\alpha-1} = M\lim_{x\to y}|x-y|^{\alpha-1}$, $M>0$

So $\frac{|f(x)-f(y)|}{|x-y|} \leq M|x-y|^{\alpha-1}$ AND f IS DIFF. SINCE THE LIMIT EXISTS.

LET $\alpha>1$. THEN $|f(x)-f(y)| \leq M|x-y|^{\alpha} \Rightarrow \frac{|f(x)-f(y)|}{|x-y|} \leq M|x-y|^{\alpha-1}$.
$(\alpha-1>0$ SINCE $\alpha>1)$.

THEN $\underbrace{\lim_{x\to y} \frac{|f(x)-f(y)|}{|x-y|}}_{\|} \leq M \cdot \lim_{x\to y} |x-y|^{\alpha-1}$

$|f'(x)| \leq M \cdot 0$ SINCE $|x-y| \to 0$ AS $x \to y$.

$\Rightarrow |f'(x)| = 0 \Rightarrow f'(x) = 0$.

$\therefore f$ IS CONSTANT SINCE $f'(x)=0$. ∎

(c) $\alpha=1$ DOES NOT NECESSARILY MEAN f IS DIFFERENTIABLE:

LET $f(x)=|x|$. $\forall x,y \in I$ $|f(x)-f(y)| = ||x|-|y|| \leq M|x-y|^{\alpha=1}$

BY PROBLEM 11.6(a), $||x|-|y|| \leq |x-y|$, SO FOR $M>0$

$\Rightarrow ||x|-|y|| < M|x-y|$.

IF $0 \in I$, $|x|$ IS NOT DIFFERENTIABLE AT $x=0$, SO $f(x)$ IS NOT NECESSARILY DIFF. ON I IF $\alpha=1$.

(d) g DIFF ON I AND g' BDD, THEN g SATISFIES A LIPSCHITZ CONDITION OF ORDER 1 ON I:

PROOF. PART (b) $\Rightarrow g$ SATISFIES A LIPSCHITZ CONDITION SINCE IT IS DIFFERENTIABLE.

g' IS BDD $\Rightarrow g' \leq M$ FOR $M>0$:

LETTING $\alpha=1$, THEN $|g(x)-g(y)| \leq M|x-y|^{\alpha=1}$, $x,y \in I$

$\Rightarrow \lim_{x\to y} \frac{|g(x)-g(y)|}{|x-y|} \leq \lim_{x\to y} M = M$

THUS $\alpha=1 \Rightarrow g' \leq M$, $M>0$

SO g SATISFIES A LIPSCHITZ CONDITION OF ORDER 1 ON I. ∎

27.6 $f(x) = g(1/x)$ $\forall x > 0$ and $L \in \mathbb{R}$. Show $\lim\limits_{x \to \infty} f(x) = L$ iff $\lim\limits_{x \to 0} g(x) = L$.

If $\lim\limits_{x \to \infty} f(x) = L$ $\forall \varepsilon > 0$ $\exists M \in \mathbb{R}^+$ such that if $x > M$ $|f(x) - L| < \varepsilon$, but $f(x) = g(\frac{1}{x})$

so $|g(\frac{1}{x}) - L| < \varepsilon$ for $x > M$.

Let $y = \frac{1}{x}$. If $x > M$, then $\frac{1}{x} < \frac{1}{M}$ $\Rightarrow y < \frac{1}{M}$. If we let $\delta = \frac{1}{M}$, we see

$|g(y) - L| < \varepsilon$ if $y < \delta$; thus

$\lim\limits_{y \to 0} g(y) = L$.

If $\lim\limits_{y \to 0} g(y) = L$, then $\forall \varepsilon > 0$ $\exists \delta > 0$ such that if $y < \delta$ then $|g(y) - L| < \varepsilon$, which implies $|g(\frac{1}{x}) - L| = |f(x) - L| < \varepsilon$. If $y < \delta$ then $x > \frac{1}{\delta}$. Setting $M = 1/\delta$, we see that if $x > M$ then $|f(x) - L| < \varepsilon$ and $\lim\limits_{x \to \infty} f(x) = L$.

28.9 (a) Let f be defined in $N(c, \varepsilon)$. Since $f''(c)$ exists by 25.17 b, we have

$$f''(c) = \lim_{h \to 0} \frac{f'(c+h) - f'(c-h)}{2h} = \lim_{h \to 0} \frac{F'(h)}{G'(h)}$$

for some functions F & G. Suppose $F(0) = 0$ and $G(0) = 0$. By L'Hopital, $\lim\limits_{h \to 0} \frac{F(h)}{G(h)} = f''(c)$.

$F(h) = f(c+h) + f(c-h) + C_1$. $F(0) = 0 \Rightarrow C_1 = -2f(c)$, $G(h) = h^2 + C_2$. $G(h) = 0 \Rightarrow C_2 = 0$ & result follows.

(b) LET $f(x) = x|x|$,

THEN $\lim_{h\to 0} \frac{f(c+h)+f(c-h)-2f(c)}{h^2} = \lim_{h\to 0} \frac{(c+h)|c+h|+(c-h)|c-h|-2c|c|}{h^2} = \frac{0}{0}$

NOTE $f(x) = x|x| = \begin{cases} x^2 & x\geq 0 \\ -x^2 & x<0 \end{cases} \Rightarrow f'(x) = \begin{cases} 2x & x\geq 0 \\ -2x & x<0 \end{cases}$

APPLYING L'HÔPITAL'S RULE TO THE LIMIT: $\lim_{h\to 0} \frac{2(c+h)+2(c-h)-4c}{2h} = \lim_{h\to 0} \frac{2c+2c-4c}{2h} = \frac{0}{0}$

APPLY L'HÔPITAL'S RULE AGAIN: $\lim_{h\to 0} \frac{2+2-4}{2} = 0$ ✓.

(IF $c<0$: $f'(c) = \lim_{h\to 0} \frac{-2(c+h)-2(c+h)+4c}{2h} = \lim_{h\to 0} \frac{-2-2+4}{2} = 0$).

BUT $f(x) = \begin{cases} x^2 & x\geq 0 \\ -x^2 & x<0 \end{cases}$,

$f'(x) = \begin{cases} 2x & x\geq 0 \\ -2x & x<0 \end{cases}$,

$f''(x) = \begin{cases} 2 & x\geq 0 \\ -2 & x<0 \end{cases}$

SINCE $f''(x)$ IS DISCONTINUOUS AT $X=0$, $f''(x)$ DOES NOT EXIST THERE.

28.12(a) $P_n(x) = f(x_0) + f'(x_0)(x-x_0) + \cdots + \frac{f^{(n)}(x_0)}{n!}(x-x_0)^n$

and $f(x) - P_n(x) = f(x) - f(x_0) - f'(x_0)(x-x_0) - \cdots - \frac{f^{(n)}(x_0)}{n!}(x-x_0)^n$

Clearly $f(x_0) - P_n(x_0) = 0$ and $(x_0 - x_0)^n = 0$

so we may apply L'Hopital's Theorem to obtain $\lim_{x\to x_0} \frac{f(x)-P_n(x)}{(x-x_0)^n} = \lim_{x\to x_0} \frac{f'(x) - f'(x_0) - \cdots - \frac{f^{(n-1)}(x_0)}{(n-1)!}(x-x_0)^{n-1}}{n(x-x_0)^{n-1}}$

Again the numerator and denominator are equal to 0 when x_0 is plugged in so Apply L'Hopital again to obtain

$\lim_{x\to x_0} \frac{f(x)-P_n(x)}{(x-x_0)^n} = \lim_{x\to x_0} \frac{f''(x) - f''(x_0) - \cdots - \frac{f^{(n-2)}(x_0)}{(n-2)!}(x-x_0)^{n-2}}{n\cdot(n-1)(x-x_0)^{n-2}}$

Continuing in this fashion, we obtain

$\lim_{x\to x_0} \frac{f(x)-P_n(x)}{(x-x_0)^n} = \lim_{x\to x_0} \frac{f^{(n-1)}(x) - f^{(n-1)}(x_0) - f^{(n)}(x_0)(x-x_0)}{n\cdot(n-1)\cdots 2\,(x-x_0)}$

$= \lim_{x\to x_0} \frac{f^{(n-1)}(x) - f^{(n-1)}(x_0)}{n!\,(x-x_0)} - \lim_{x\to x_0} \frac{f^{(n)}(x_0)(x-x_0)}{n!\,(x-x_0)}$

$= \frac{f^{(n)}(x_0)}{n!} - \frac{f^{(n)}(x_0)}{n!} = 0$

(b) Since we don't know if $f^{(n+1)}(x_0)$ exists we can not apply n derivatives. If we did we would have an expression involving $f^{(n+1)}(x_0)$.

$\boxed{29.9}$ f, g ARE INTEGRABLE ON $[a,b]$ & h IS A FUNCTION S.T. $f(x) \leq h(x) \leq g(x)$ $\forall x \in [a,b]$. THEN h IS INTEGRABLE.

COUNTEREXAMPLE: LET $f(x) = 0$ AND $g(x) = 1$ $\forall x \in [a,b]$.

LET $h(x) = \begin{cases} 1 & x \text{ IS RATIONAL} \\ 0 & x \text{ IRRATIONAL} \end{cases}$

WE SEE THAT $f(x) = 0 \leq h(x) \leq 1 = g(x)$ $\forall x \in [a,b]$,

BUT $h(x)$ IS NOT INTEGRABLE ON $[a,b]$ (EXAMPLE 29.8).

$\boxed{29.11}$ ⓐ f IS BOUNDED ON $[a,b]$ & SUPPOSE $\exists$ A SEQUENCE P_n OF PARTITIONS S.T. $\lim_{n\to\infty} U(f,P_n) - L(f,P_n) = 0$. THEN f IS INTEGRABLE.

29.11(a) since $\lim_{n\to\infty} U(f, P_n) - L(f, P_n) = 0$, $\forall \varepsilon > 0$ $\exists N$ such that $\forall n > N$ $|U(f, P_n) - L(f, P_n)| < \varepsilon$

But $U(f, P_n) \geq L(f, P_n) \Rightarrow$ $U(f, P_n) - L(f, P_n) < \varepsilon$ and by the theorem stating f is integrable iff $\forall \varepsilon > 0$ $\exists$ P of $[a, b]$ with $U(f, P) - L(f, P) < \varepsilon$ we see f is integrable

$\boxed{29.13}$ f IS CONTIN. ON $[a,b]$. & $f(x) \geq 0$ $\forall x \in [a,b]$. IF $L(f) = 0$ THEN $f(x) = 0$ $\forall x \in [a,b]$.

PROOF. LET P BE A PARTITION OF $[a,b]$.

$L(f) = 0 \Rightarrow \sup\{L(f,P) \mid P \text{ IS A PARTITION OF } [a,b]\} = 0$ BY DEF.

SO $L(f) = 0 = \sup\{\sum_{i=1}^{n} m_i \Delta x_i \mid m_i = \inf\{f(x) \mid x \in [x_{i-1}, x_i]\}\}$.

$\Rightarrow \sum_{i=1}^{n} m_i \Delta x_i \leq 0$ FOR EACH $[x_{i-1}, x_i]$.

use with $f(x) \geq 0$ $\forall x$

f IS INTEGRABLE MEANS $L(f) = U(f)$, SO $U(f) = 0$. (SO $\int_a^b f(x)\,dx = 0$)

$\Rightarrow 0 = \inf\{\sum_{i=1}^{n} M_i \Delta x_i \mid M_i = \sup\{f(x) \mid x \in [x_{i-1}, x_i]\}\}$.

$\Rightarrow U(f,P) = \sum_{i=1}^{n} M_i \Delta x_i \geq 0$. no, this is assumed

SINCE $U(f,P) \geq L(f,P)$ $\boxed{\Rightarrow L(f,P) = 0}$.

BUT THIS MEANS $U(f,P) = 0$ SINCE f IS CONTINUOUS

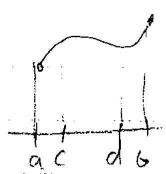
$\Rightarrow f(x) = 0$ $\forall x \in [a,b]$. ∎

$\boxed{29.15}$ f IS INTEGRABLE ON $[a,b]$ & $[c,d] \subseteq [a,b]$. THEN f IS INTEGRABLE ON $[c,d]$.

PROOF. IF $[c,d]=[a,b]$, WE'RE DONE.

SO ASSUME $[c,d] \subsetneq [a,b]$.

BY THM 29.9, f IS INTEGRABLE $\Rightarrow \forall \varepsilon>0 \ \exists$ A PARTITION P OF $[a,b]$ S.T. $U(f,P)-L(f,P)<\varepsilon$.



WE KNOW $U(f)=L(f)$. LET $P=\{x_0=a, x_1, \dots, \underbrace{x_r=c, \dots, x_s=d}_{\text{CALL THIS } P_{[c,d]};\ P_{[c,d]} \subsetneq P}, \dots, x_n=b\}$ FOR A GIVEN ε. THEN $L(f,P)=\sum_{i=1}^{n} m_i \Delta x_i$ AND $U(f,P)=\sum_{i=1}^{n} M_i \Delta x_i$.

$U(f)=\inf\{U(f,P) \mid P \text{ IS A PARTITION OF } [a,b]\}=\sup\{L(f,P) \mid P \text{ PARTITION}\}=L(f)$

$\Rightarrow \sum_{i=r}^{s} m_i \Delta x_i \le \sum_{i=1}^{n} m_i \Delta x_i = L(f,P)$ AND

$\sum_{i=r}^{s} M_i \Delta x_i \le \sum_{i=1}^{n} M_i \Delta x_i = U(f,P)$.

$\sum_{i=r}^{s} m_i \Delta x_i = L(f,P_{[c,d]})$ AND $\sum_{i=r}^{s} M_i \Delta x_i = U(f,P_{[c,d]})$.

SO $L(f,P_{[c,d]}) \le L(f) = U(f) \ge U(f,P_{[c,d]})$.

THEN $U(f,P)-L(f,P)<\varepsilon \Rightarrow U(f,P_{[c,d]})-L(f,P_{[c,d]})<\varepsilon$

$\Rightarrow$ f IS INTEGRABLE ON $[c,d]$ BY THM 29.9. ∎

$\boxed{30.5}$ f IS CONTINUOUS ON $[a,b]$ AND $f(x)\ge 0 \ \forall x \in [a,b]$. IF $\exists c \in [a,b]$ S.T. $f(c)>0$, THEN $\int_a^b f(x)\,dx > 0$.

30.5 Suppose $\exists c \in [a,b]$ s.t. $f(c)>0$. Since $f(c)$ is continuous by a previous homework $\exists \alpha < f(c)$ such that $f(x)>\alpha$ on $N(c;\delta)$ some δ. Choose a partition P such that $x_{i+1}-x_i<\delta \ \forall i$. Note $c \in [x_i, x_{i+1}]$ for some i and $m_i \ge \alpha$ thus $L(f,P) \ge m_i \Delta x_i > 0$. Since f is continuous, then f is integrable and $\int_a^b f(x)\,dx = L(f,P) > 0$.

$\boxed{30.10}$ f IS NOT INTEGRABLE, BUT $|f|$ IS: $\quad (f: [0,1] \to \mathbb{R})$

LET $f(x) = \begin{cases} 1 & x \text{ IS RAT'L} \\ -1 & x \text{ IRRAT'L} \end{cases}$ (VERY SIMILAR TO DIRICHLET FUNCTION).

WE KNOW f IS NOT INTEGRABLE $\forall x \in [0,1]$.

BUT $|f(x)| = \begin{cases} |1| & x \text{ RAT'L} \\ |-1| & x \text{ IRRAT'L} \end{cases} = \begin{cases} 1 & x \text{ RAT'L} \\ 1 & x \text{ IRRAT'L} \end{cases} = 1$ IS INTEGRABLE SINCE IT IS CONTINUOUS (THM 30.2).

$\boxed{30.11}$ f IS INTEGRABLE ON $[a,b]$ AND $m \le f(x) \le M$ $\forall x \in [a,b]$. SHOW THAT $m(b-a) \le \int_a^b f(x)\,dx \le M(b-a)$.

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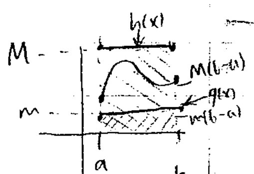

PROOF. LET $g(x) = m$ AND $h(x) = M$ $\forall x \in [a,b]$.

THEN WE HAVE $m = g(x) \le f(x) \le M = h(x)$.

BY THM 30.2, g AND h ARE INTEGRABLE SINCE THEY ARE CONTINUOUS.

$\Rightarrow \int_a^b g(x)\,dx \le \int_a^b f(x)\,dx \le \int_a^b h(x)\,dx$ (PRACTICE 30.5).

BUT $g(x) = m$, $h(x) = M$, SO WE HAVE $\int_a^b m\,dx \le \int_a^b f(x)\,dx \le \int_a^b M\,dx$.

BY RULES OF INTEGRATION, $\int_a^b m\,dx = m(x|_a^b)$, AND $\int_a^b M\,dx = M(x|_a^b)$

SO $m(b-a) \le \int_a^b f(x)\,dx \le M(b-a)$. ■

$\boxed{30.21}$ CAUCHY-SCHWARZ INEQUALITY: IF f, g ARE INTEGRABLE ON $[a,b]$, THEN $\left(\int_a^b fg\right)^2 \le \left(\int_a^b f^2\right)\left(\int_a^b \right.$

PROOF. LET $\alpha = \int_a^b g^2$ AND $-\beta = \int_a^b fg$ AND CONSIDER $\int_a^b (\alpha f + \beta g)^2$.

$-\beta^2 \le (\int_a^b f^2)\alpha$

30.21 Let $\alpha = \int_a^b g^2(x)dx$ and $\beta = -\int_a^b f(x)g(x)dx$

Since f and g are integrable,

$\alpha f(x) + \beta g(x)$ is integrable, $f^2(x)$, $g^2(x)$ and $(\alpha f(x) + \beta g(x))^2$ are integrable.

Note that

$$0 \le \int_a^b (\alpha f(x) + \beta g(x))^2 dx = \alpha^2 \int_a^b f^2(x)dx + 2\alpha\beta \int_a^b f(x)g(x)dx + \beta^2 \int_a^b g^2(x)dx$$

$$= \left(\int_a^b g^2(x)dx\right)^2 \int_a^b f^2(x)dx - 2\int_a^b g^2(x)dx \left(\int_a^b f(x)g(x)dx\right)^2 + \left(\int_a^b f(x)g(x)dx\right)^2 \int_a^b g^2(x)dx$$

$$= \int_a^b g^2(x)dx \left(\int_a^b f^2(x)dx \int_a^b g^2(x)dx - \left(\int_a^b f(x)g(x)dx\right)^2\right)$$

and $\int_a^b g^2(x)dx \ge 0$

Thus $\int_a^b f^2(x)dx \int_a^b g^2(x)dx \ge \left(\int_a^b f(x)g(x)dx\right)^2$