26.6 Three Functions:
1) $\sin(x)$ on $[0, 1]$. This function is continuous and differentiable on $[0, 1]$, but $\sin(0) \neq \sin(1)$. There is no $c \in (0, 1)$ s.t. $\frac{\Delta}{\Delta x} \sin(x) = 0$.

2) $|x|$ on $[-5, 5]$. This function is continuous and $|-5| = 5$, but $|x|$ is not differentiable at $x = 0$. So, there is no $c \in (-5, 5)$ s.t. $\frac{\Delta}{\Delta x} |x| = 0$.

3) $\frac{1}{|x|}$ on $[1, 0) \cup (0, 1]$. This function is differentiable and $\frac{\Delta}{\Delta x} \frac{1}{|x|} = \frac{1}{x^2}$, but it is not continuous. There is no $c \in (0, 1)$ s.t. $\frac{\Delta}{\Delta x} \frac{1}{|x|} = 0$.

26.8 If $f$ is increasing on $I$, then $f'(x) \geq 0 \forall x \in I$.

Proof. Suppose $f$ is increasing on $I$. Then $x_i < x_j \Rightarrow f(x_i) \leq f(x_j)$ $\forall i < j$.

Since $f$ is differentiable, $f'(x_j) = \frac{f(x_j) - f(x_i)}{x_j - x_i}$.

But $f(x_i) \leq f(x_j)$ $\forall x_i < x_j$, so $f'(x_j) \geq 0$ $\forall j > i$.

$f'(x) = \frac{f(x) - f(x_j)}{x - x_j} \geq 0$ $\forall j \geq 0$ since $x > x_i$.

$\Rightarrow f'(x) \geq 0 \forall x \in I$.

Now suppose $f'(x) \geq 0 \forall x \in I$.

By MVT, $\exists c \in I$ s.t. $f(x_j) - f(x_i) = f'(c)(x_j - x_i)$, $x_i < x_j$.

If $f'(c) = 0$, $f(x_i) = f(x_j)$.

If $f'(c) > 0$, then by Thm 26.8 $f(x_j) > f(x_i)$.

$\Rightarrow f(x) \geq f(x_i) \forall x_i < x_j \in I$, so $f$ is increasing on $I$.

26.8 If $f$ is decreasing on $I$, then $f'(x) \leq 0 \forall x \in I$.

Proof. If $f$ is decreasing on $I$ then $x_i < x_j \Rightarrow f(x_i) \geq f(x_j)$ $\forall i < j$.

Since $f$ is differentiable, $f'(x_j) = \frac{f(x_j) - f(x_i)}{x_j - x_i}$.

But $f(x_i) \geq f(x_j)$ $\forall x_i < x_j$, so $f'(x_j) \leq 0$ $\forall j > i$.

$f'(x_j) = \frac{f(x_j) - f(x_i)}{x_j - x_i} \geq 0$ $\forall i < j$ $\Rightarrow f'(x) \leq 0 \forall x \in I$.

Now suppose $f'(x) \leq 0 \forall x \in I$.

By MVT, $\exists c \in I$ s.t. $f(x_j) - f(x_i) = f'(c)(x_j - x_i)$, $x_i < x_j$.

If $f'(c) = 0$, $f(x_i) = f(x_j)$.

If $f'(c) < 0$, then by Thm 26.8, $f(x_i) < f(x_j)$.
\[ f(x) \geq f(y) \quad \forall \ x < y, \ i.e., \ f \text{ is decreasing}. \]

**26.18** \( f \) IS DIFF. ON \((a,b)\) \( \iff \) CONT. ON \([a,b] \) WITH \( f(a) = f(b) = 0 \). Prove that
\[ \forall \ k \in \mathbb{R} \ \exists \ c \in (a,b) \text{ s.t. } f'(c) = k f(c), \text{ where } g(x) = e^{-kx} f(x). \]

*Proof.* Since \( f \) is CONTINUOUS on \([a,b]\) and DIFFERENTIABLE on \((a,b)\), and \( e^{-kx} \) is CONTINUOUS as well, then \( g(x) = e^{-kx} f(x) \) is CONTINUOUS on \([a,b]\) and DIFF. ON \((a,b)\).

Because \( f(a) = f(b) = 0 \), \( g(a) = g(b) = 0 \).

By Rolle's thm \( \Rightarrow \exists c \in (a,b) \) s.t. \( g'(c) = 0 \).

Since \( g'(x) = -ke^{-kx} f(x) + e^{-kx} f'(x) \) (Product rule for derivatives),
\[ g'(c) = 0 = -ke^{-kc} f(c) + e^{-kc} f'(c), \]
\[ \Rightarrow \ e^{kc} f'(c) = k e^{kc} f(c), \]

Concluding out the \( e^{kc} \) term on both sides yields
\[ f'(c) = kf(c). \]

This holds \( \forall k \in \mathbb{R} \) since \( k \) was chosen arbitrarily. \( \blacksquare \)

**26.23** \( f \) IS DEFINED ON \( I \), AND SUPPOSE \( \exists M, a > 0 \) s.t. \( |f(x) - f(y)| \leq M \left| x - y \right|^a \ \forall x, y \in I. \)

\( \circ \) \( f \) IS UNIFORMLY CONTINUOUS ON \( I. \)

*Proof.* \( f \) IS UNIF. CONT. MEANS \( \exists \delta > 0 \) \( \exists \varepsilon > 0 \) \( \forall x, y \in I \) \( |f(x) - f(y)| < \varepsilon \) whenever \( |x - y| < \delta \).

Let \( \delta = \frac{\varepsilon}{M^{1/a}} \). \( \exists \delta > 0 \) \( \forall x, y \in I \) \( |f(x) - f(y)| < \varepsilon \) whenever \( |x - y| < \delta \).

Thus, we have that \( |f(x) - f(y)| \leq M |x - y|^a \leq M \left| x - y \right|^a \leq \varepsilon \).

So \( f \) IS UNIF. CONTINUOUS. \( \blacksquare \)

\( \circ \) \( a > 1 \Rightarrow f \) IS CONSTANT.

*Proof.* \( f \) IS DIFFERENTIABLE:

DEF. OF DIFFERENTIABLE: \( \lim_{x \to y} \frac{f(x) - f(y)}{x - y} \) EXISTS.

Setting \( c = y \), we have that \( \frac{f(x) - f(y)}{x - y} \to \partial_x f(y) \) \( x, y \in I. \)

But \( \lim_{x \to y} \frac{f(x) - f(y)}{x - y} \leq \lim_{x \to y} \frac{M |x - y|^a}{|x - y|} = M \lim_{x \to y} |x - y|^{a - 1} \leq \varepsilon \) \( x, y \in I. \)
So \( \frac{|f(y) - f(x)|}{|x-y|^{d-1}} \leq M |x-y|^{d-1} \) and \( f \) is differentiable since the limit exists.

Let \( d > 1 \). Then \( |f(x) - f(y)| \leq M |x-y|^{d-1} \Rightarrow \frac{|f(x) - f(y)|}{|x-y|} \leq M |x-y|^{d-1} \).

\( d > 1 \) since \( d > 1 \).

\[ \frac{\|f(x) - f(y)\|}{|x-y|} \leq M |x-y|^{d-1} \]

\[ f'(x) \leq M \quad \text{since} \quad |x-y| \to 0 \quad \text{as} \quad x \to y. \]

\[ \Rightarrow |f'(x)| = 0 \Rightarrow f'(x) = 0. \]

\[ \therefore \ f \text{ is constant since } f'(x) = 0. \]

\( \square \)

\( \alpha = 1 \) does not necessarily mean \( f \) is differentiable.

Let \( f(x) = |x| \). \( \forall x, y \in I \quad |f(x) - f(y)| = |x - y| \leq M |x-y|^{d=1} \).

By Problem 11.6(a), \( ||x|-|y|| \leq |x-y| \), so for \( M > 0 \)

\[ |x-y| < M |x-y|^{d} \]

If \( 0 \in I \), \( |x| \) is not differentiable at \( x = 0 \), so

\[ f(x) \text{ is not necessarily differentiable on } I \text{ if } \alpha = 1. \]

\( g \) different on \( I \) and \( g' \) bdd., then \( g \) satisfies a Lipschitz condition of order 1 on \( I \):

Proof: \( \alpha = 1 \Rightarrow g \) satisfies a Lipschitz condition since it is differentiable.

\( g' \) is bdd \( \Rightarrow g' \leq M \) for \( M > 0 \);

letting \( \alpha = 1 \), then \( |g(x) - g(y)| \leq M |x-y|^{d=1}, x, y \in I \)

\[ \Rightarrow \frac{|g(x) - g(y)|}{|x-y|} \leq M \quad M = M \]

Thus \( \alpha = 1 \Rightarrow g' \leq M \), \( M > 0 \)

So \( g \) satisfies a Lipschitz condition of order 1 on \( I \). \( \square \)
27.6 \(f(x) = g(y(x)) \forall x \geq 0 \text{ and } \forall x \geq 0. \text{ Show lim}_{x \to \infty} f(x) = L \iff \lim_{x \to \infty} g(x) = L.\)

If \(\lim_{x \to \infty} f(x) = L\), \(\forall x \geq 0 \exists M \in \mathbb{R}^+\) such that
\(\text{if } x > M \implies |f(x) - L| < \varepsilon, \text{ but } f(x) = g(y(x))\)
so \(|g(y(x)) - L| < \varepsilon \text{ for } x > M.\)

Let \(y = \frac{1}{x}\). If \(x > M\), then \(\frac{1}{x} < \frac{1}{M}\)
\(\Rightarrow y < \frac{1}{M}\). If we let \(s = \frac{1}{M}\), we see
\(|g(y) - L| < \varepsilon \iff y < \delta; \text{ thus}\)
\(\lim_{y \to 0} g(y) = L.\)

If \(\lim_{y \to 0} g(y) = L\), then \(\forall \varepsilon > 0 \exists \delta > 0\)
such that \(\text{if } y < \delta \text{ then } |g(y) - L| < \varepsilon\), which implies
\(|f(x) - L| = |g(y(x)) - L| < \varepsilon. \text{ If } y < \delta\)
\(\text{then } x > \frac{1}{s}; \text{ setting } M = \frac{1}{s}, \text{ we see that}\)
\(\text{if } x > M \text{ then } f(x) - L < \varepsilon\)
\(\text{and } \lim_{x \to \infty} f(x) = L.\)

28.9 (a) \(\text{Let } f \text{ be defined in } N(0, \varepsilon). \text{ Since } f''(c) \text{ exists by } 25.17\), we have
\(f''(c) = \lim_{h \to 0} \frac{f(c+h) - f(c-h)}{2h} = \lim_{h \to 0} G(h)\)
for some functions \(F \& G). \text{ Suppose } F(0) = 0 \text{ and } G(0) = 0. \text{ By } L'Hopital, \lim_{h \to 0} G(h) = 0.\)
\(F(h) = f(c+h) + f(c-h) + C \left( \lim_{h \to 0} f(h) = f(0) \right) \Rightarrow\)
\(C = f(0) \quad \text{and} \quad G(h) = h^2 - c_2 \Rightarrow G(h) = 0 \Rightarrow c_2 = 0.\)
28.12(a) \[ P_n(x) = f(x_0) + f'(x_0)(x-x_0) + \cdots + \frac{f^{(n)}(x_0)}{n!}(x-x_0)^n \]

and \[ f(x) - P_n(x) = f(x) - f(x_0) - f'(x_0)(x-x_0) - \cdots - \frac{f^{(n)}(x_0)}{n!}(x-x_0)^n \]

Clearly, \[ f(x_0) - P_n(x_0) = 0 \] and \[ (x-x_0)^n = 0 \]

So we may apply L'Hôpital's Theorem to

\[ \lim_{x \to x_0} \frac{f(x) - P_n(x)}{(x-x_0)^n} = \lim_{x \to x_0} \frac{f'(x) - f'(x_0) - \cdots - \frac{f^{(n)}(x_0)}{n!}(x-x_0)^{n-1}}{n(x-x_0)^{n-1}} \]

Again the numerator and denominator are equal to 0 when \( x_0 \) is plugged in. So apply L'Hôpital again to obtain

\[ \lim_{x \to x_0} \frac{f(x) - P_n(x)}{(x-x_0)^n} = \lim_{x \to x_0} \frac{f''(x) - f''(x_0) - \cdots - \frac{f^{(n-1)}(x_0)}{(n-1)!}(x-x_0)^{n-2}}{n(n-1)!x-x_0)^{n-2}} \]

Continuing in this fashion, we obtain

\[ \lim_{x \to x_0} \frac{f(x) - P_n(x)}{(x-x_0)^n} = \lim_{x \to x_0} \frac{f^{(n-1)}(x) - f^{(n-1)}(x_0) - \cdots - \frac{f^{(n)}(x_0)}{n!(n-1)!}(x-x_0)}{n(n-1)!x-x_0)^{n-2}} \]

\[ = \lim_{x \to x_0} \frac{f^{(n-1)}(x) - f^{(n-1)}(x_0)}{n!(x-x_0)} - \lim_{x \to x_0} \frac{f^{(n)}(x_0)(x-x_0)}{n!(x-x_0)} \]

\[ = \frac{f^{(n)}(x_0)}{n!} - \frac{f^{(n)}(x_0)}{n!} = 0 \]

(b) Since we don't know if \( f^{(n+1)}(x_0) \) exists, we cannot apply \( n \) derivatives. If we did, we would have an expression involving \( f^{(n+1)}(x_0) \).
29.9 Let $f, g$ be integrable on $[a, b]$ and $h$ is a function s.t. $0 \leq h(x) \leq g(x) \forall x \in [a, b]$. Then $h$ is integrable.

**Counterexample:** Let $f(x) = 0$ and $g(x) = 1 \forall x \in [a, b]$.

Let $h(x) = \begin{cases} 1 & x \text{ is rational} \\ 0 & x \text{ is irrational} \end{cases}$.

We see that $f(x) = 0 \leq h(x) \leq 1 = g(x), \forall x \in [a, b]$.

But $h$ is not integrable on $[a, b]$ (example 29.8).

29.11 Suppose $f$ is bounded on $[a, b]$ and $f$ is integrable on $[a, b]$ if $\lim_{n \to \infty} u(P_n) - l(P_n) = 0$. Then $f$ is integrable.

29.11(a) Since $\lim_{n \to \infty} u(f, P_n) - L(f, P_n) = 0, \forall \epsilon > 0 \exists N$ such that $\forall n > N |u(f, P_n) - L(f, P_n)| < \epsilon$.

But $u(f, P_n) - L(f, P_n) < \epsilon$ and by Theorem stating $f$ is integrable if $\forall \epsilon > 0 \exists P$ of Eq 6.7 with $u(f, P) - L(f, P) < \epsilon$,

we see $f$ is integrable.

29.12 $f$ is continuous on $[a, b]$. If $f(x) \geq 0 \forall x \in [a, b]$. If $L(f) = 0$ then $f(x) = 0 \forall x \in [a, b]$.

**Proof:** Let $P$ be a partition of $[a, b]$.

$L(f) = 0 \Rightarrow \sup \{L(f, P) | P \text{ is a partition of } [a, b]\} = 0$ by def.

So $L(f) = 0 = \sup \{\sum_{i=1}^{n} M_i \Delta x_i | M_i = \inf \{f(x) | x \in [x_{i-1}, x_i]\}\}$

$\Rightarrow \sum_{i=1}^{n} M_i \Delta x_i \leq 0$ for each $[x_{i-1}, x_i]$.

$f$ is integrable means $L(f) = U(f), \Rightarrow U(f) = 0$.

$\Rightarrow u(f, P) = \sum_{i=1}^{n} M_i \Delta x_i \geq 0$.

Since $U(f, P) = L(f, P)$, so $L(f, P) = 0$ since $f$ is continuous.

But this means $u(f, P) = 0$. Since $f$ is continuous,

$\Rightarrow f(x) = 0 \forall x \in [a, b]$. $\blacksquare$
29.5 If \( f \) is integrable on \([a, b]\) \( \subseteq \) \( [c, d] \subseteq [a, b] \), then \( f \) is integrable on \([c, d]\).

**Proof.** If \([c, d] = [a, b]\), we're done.

So assume \([c, d] \subsetneq [a, b]\).

By Thm 29.9, \( f \) is integrable \( \Rightarrow \) \( \forall \varepsilon > 0 \exists \text{ a partition } P \) of
\([a, b]\) s.t. \( U(f, P) - L(f, P) < \varepsilon \).

We know \( U(f) = L(f) \).

Let \( P = \{x_0 = a, x_1, \ldots, x_i, \ldots, x_k = b\} \) for a given \( \varepsilon \).

Then \( U(f, P) = \sum_{i=1}^{k} M_i \Delta x_i \) and
\( U(f) = \sum_{i=1}^{k} M_i \Delta x_i \).

\( \Rightarrow \) \( \sum_{i=1}^{n} m_i \Delta x_i \leq \sum_{i=1}^{n} M_i \Delta x_i = L(f, P) \)
\( \Rightarrow \) \( \sum_{i=1}^{n} m_i \Delta x_i \leq L(f, P) \).

\( \sum_{i=1}^{n} m_i \Delta x_i = U(f, P) \) and \( \sum_{i=1}^{n} M_i \Delta x_i = U(f) \).

Thus \( L(f, P) - U(f, P) < 0 \Rightarrow U(f, P) < L(f) \Rightarrow U(f) < L(f) \).

By Thm 29.9, \( f \) is integrable on \([c, d]\).

30.5 \( f \) is continuous on \([a, b]\) and \( f(x) \geq 0 \) \( \forall x \in [a, b] \). If \( \exists \epsilon \in [a, b] \) s.t. \( f(\epsilon) > 0 \), then \( \int_{a}^{b} f(x) \, dx > 0 \).

**Suppose** \( \exists c \in [a, b] \) s.t. \( f(c) > 0 \). Since \( f(c) \) is continuous by a previous homework, \( \exists \delta < f(c) \) such that \( f(x) > \delta \) on \( N(c; \delta) \) some \( \delta \).

Choose a partition \( P \) such that \( x_{i+1} - x_i < \delta \) \( \forall i \).

Note \( c \in [x_i, x_{i+1}] \) for some \( i \) and \( m_i \geq \delta \) thus \( L(f, P) \geq m_i \Delta x_i > 0 \). Since \( f \) is continuous, then \( f \) is integrable and \( \int_{a}^{b} f(x) \, dx = L(f, P) > 0 \).
\[ 30.10 \quad f \text{ is not integrable, but } |f| \text{ is: } (f : [0,1] \rightarrow \mathbb{R}) \]

Let \( f(x) = \begin{cases} \frac{1}{x} & \text{if } x \in \mathbb{R}^+ \setminus \mathbb{Q} \\ -1 & \text{if } x \in \mathbb{Q} \end{cases} \) (very similar to Dirichlet function).

We know \( f \) is not integrable. \( \forall \epsilon > 0 \), \( \exists \eta > 0 \) such that \( |f(x)| \geq \epsilon \) \( \forall x \in [a,b] \setminus [a,b] \). 

But \( |f(x)| = \begin{cases} \frac{1}{x} & \text{if } x \in \mathbb{R}^+ \setminus \mathbb{Q} \\ 1 & \text{if } x \in \mathbb{Q} \end{cases} \) \( \exists \xi \) such that \( |f(\xi)| = 1 \) is integrable since it is continuous (Theorem 30.2).

\[ 30.11 \quad f \text{ is integrable on } [a,b] \text{ and } m \leq f(x) \leq M \forall x \in [a,b]. \text{ Show that} \]
\[
m(b-a) \leq \int_a^b f(x) \, dx \leq M(b-a). \]

\[ 5/6 \]

Proof. Let \( g(x) = m \) and \( h(x) = M \forall x \in [a,b] \).

Then we have \( m \leq g(x) \leq f(x) \leq h(x) \).

By Theorem 30.2, \( g \) and \( h \) are integrable since they are continuous.

\[
\Rightarrow \int_a^b g(x) \, dx \leq \int_a^b f(x) \, dx \leq \int_a^b h(x) \, dx \quad \text{(Practice 30.5).}
\]

But \( g(x) = m, \ h(x) = M \), so we have \( \int_a^b m \, dx \leq \int_a^b f(x) \, dx \leq \int_a^b M \, dx \).

By Rules of Integration, \( \int_a^b m \, dx = m(b-a) \), and \( \int_a^b M \, dx = M(b-a) \).

So \( m(b-a) \leq \int_a^b f(x) \, dx \leq M(b-a). \)

\[ 30.24 \quad \text{Cauchy-Schwarz Inequality: If } f, g \text{ are integrable on } [a,b], \text{ then } \left( \int_a^b f(x)g(x) \right)^2 \leq \left( \int_a^b f(x)^2 \right) \left( \int_a^b g(x)^2 \right). \]

Proof. Let \( \alpha = \int_a^b g(x)^2 \) and \( -\beta = \int_a^b f(x)^2 \) and consider \( \int_0^1 (\alpha + \beta + \alpha \beta) \).

\[ \beta \leq (\beta + \alpha)^2 \]

\[ 2 \]

\[ \alpha \]

\[ 4 \]

\[ 6 \]
Let \( \alpha = \int_a^b g^2(x) \, dx \) and \( \beta = -\int_a^b f(x)g(x) \, dx \).

Since \( f \) and \( g \) are integrable,
\[ \alpha f(x) + \beta g(x) \] is integrable, \( f^2(x), g^2(x) \) and \( (\alpha f(x) + \beta g(x))^2 \) are integrable.

Note that
\[
0 \leq \int_a^b (\alpha f(x) + \beta g(x))^2 \, dx = \alpha^2 \int_a^b f^2(x) \, dx + 2\alpha \beta \int_a^b f(x)g(x) \, dx + \beta^2 \int_a^b g^2(x) \, dx
\]
\[
= \left(\int_a^b g^2(x) \, dx\right)^2 \int_a^b f^2(x) \, dx
\]
\[
- 2 \int_a^b g^2(x) \, dx \left(\int_a^b f(x)g(x) \, dx\right)^2
\]
\[
+ \left(\int_a^b f(x)g(x) \, dx\right)^2 \left(\int_a^b g^2(x) \, dx\right)^2
\]
\[
= \int_a^b g^2(x) \, dx \left(\int_a^b f^2(x) \, dx - \frac{\left(\int_a^b f(x)g(x) \, dx\right)^2}{\left(\int_a^b g^2(x) \, dx\right)^2}\right)
\]
and \( \int_a^b g^2(x) \, dx \geq 0 \).

Thus, \( \int_a^b f^2(x) \, dx \geq \left(\int_a^b f(x)g(x) \, dx\right)^2 \).