Answer keys for the 1st week homework

4(p.8). When \( a = tb \), for any \( t \in \mathbb{R} \).

5(p.8). When \( a = tb \), for any \( t \in \mathbb{R}_+ \).

6(p.12). Since \( \overline{S} = S \cup \partial S \) and \( \partial S = \partial(S^c) \), we have
\[ S^c = S^c \cup \partial(S^c) = S^c \cup \partial S. \]
Thus,
\[ S \cap S^c = (S \cup \partial S) \cap (S^c \cup \partial S) = (S \cap S^c) \cup \partial S = \emptyset \cap \partial S = \partial S. \]

8(p.19). Fix \( x_0 \in \mathbb{R}^n \) and \( \epsilon > 0 \), let us consider a open ball in \( \mathbb{R}^k \),
\[ B(\epsilon, f(x_0)) = \{ y \in \mathbb{R}^k : |y - f(x_0)| < \epsilon \}. \]
Then, by assumption, \( \{ x : f(x) \in B(\epsilon, f(x_0)) \} \) is open. Hence there exists \( \delta > 0 \) such that \( \overline{x} \in \{ x : f(x) \in B(\epsilon, f(x_0)) \} \) as soon as \( |x_0 - \overline{x}| < \delta \). But if \( \overline{x} \in \{ x : f(x) \in B(\epsilon, f(x_0)) \} \) then \( f(\overline{x}) \in B(\epsilon, f(x_0)) \), that is \( |f(x_0) - f(\overline{x})| < \epsilon \). Thus, we can conclude that the function \( f \) is continuous. Using the Proposition 1.4 (p.10), one can similarly prove that the same result holds if open is replaced by closed.