Answer keys for the 2nd week homework

5(p.23). Suppose that $\lim_{\mathbf{x}\to\mathbf{a}} f(x) = l$. Then for any number $\epsilon > 0$ there is a number $\delta > 0$ so that

$$|f(\mathbf{x}) - l| < \epsilon \text{ whenever } 0 < |\mathbf{x} - \mathbf{a}| < \delta\,.$$

We now consider any sequence $\{\mathbf{x}_k\}$ that converges to $\mathbf{a}$. Then, by definition of the convergent sequence, for any $\delta > 0$ there is an integer K such that $|\mathbf{x}_k - \mathbf{a}| < \delta$ if $k > K$. By combining these two statements, we have for any $\epsilon > 0$ there is an integer K such that $|f(\mathbf{x}_k) - l| < \epsilon$ whenever $k > K$. Conversely, we assume that, for every sequence $\{x_k\}$ that converges to $\mathbf{a}$, $f(\mathbf{x}_k) \to l$ and $\lim_{\mathbf{x}\to\mathbf{a}} f(x) \neq l$. Since $\lim_{\mathbf{x}\to\mathbf{a}} f(x) \neq l$, for all $\delta > 0$ there is $\epsilon > 0$ and $\mathbf{x}$ such that $|\mathbf{x} - \mathbf{a}| < \delta$ but $|f(\mathbf{x}) - l| \geq \epsilon$. Taking $\delta = 1/k$, we can always find the points $\mathbf{x}_k$'s so that $|\mathbf{x}_k - \mathbf{a}| < 1/k$ but $|f(\mathbf{x}_k) - l| \geq \epsilon$. This contradict to our another assumption $f(\mathbf{x}_k) \to l$.

3(p.28). Let us consider $a \in [0,1]$ is a irrational number. Otherwise, it is clear that there is an subsequence that converges to a. For any irrational number $a \in [0,1]$, consider its decimal expansion, that is,

$$a = 0.d_1d_2d_3d_4... = \sum_{n=1}^{\infty} \frac{d_n}{10^n}\,,$$

where $d_n \in \{0,1,2,...,8,9\}$, for all n. Let us define the sequence

$$a_k = \sum_{n=1}^{k} \frac{d_n}{10^n}\,.$$

Then one can easily check that the sequence $\{a_k\}_{k=1}^{\infty}$ is a subsequence of the given sequence and $\{a_k\}_{k=1}^{\infty}$ converges to a .

7(p.29). Let us assume that there is a subsequence of $\{\mathbf{x}_k\}$, $\{\mathbf{x}_{k_j}\}_{j=1}^{\infty}$, converges to $\mathbf{x}$. For any given number r, consider the ball centered at $\mathbf{x}$ with radius r, $B(r,\mathbf{x})$. Taking $\epsilon = r$ in the definition of the *convergent sequence*, for every $r > 0$ there is an integer J_r such that $|\mathbf{x} - \mathbf{x}_{k_j}| < r$ if $j > J_r$. Thus, there are infinitely many values of k so that every ball centered at $\mathbf{x}$ contained $\mathbf{x}_k$. we now assume conversely, i.e. every ball centered at $\mathbf{x}$ contains $\mathbf{x}_k$ for infinitely many values of k. For some fixed number r, consider the ball $B(r,\mathbf{x})$. Then by assumption there are infinitely many $\mathbf{x}_k$'s which contained in $B(r,\mathbf{x})$. Thus one can always pick any point inside the ball. Let us pick any point in the ball and set $\mathbf{x}_{k_1}$ Again, we consider the ball $B(r/2,\mathbf{x})$ and pick a point in the ball $B(r/2,\mathbf{x})$ and set $\mathbf{x}_{k_2}$ By repeating this process (pick a point in $B(r/2^j,\mathbf{x})$ and set x_{k_j}), one can built the subsequence of $\{\mathbf{x}_k\}$, $\{\mathbf{x}_{k_j}\}$, which converge to $\mathbf{x}$.

5(p.33). Let $G_k = (S_k)^c$. Assume that there is no point contained in all of S_k's, that is , $\bigcap_1^{\infty} S_k = \emptyset$. Then the sets

$$\{G_k\}_2^{\infty} = \{G_2, G_3, \cdots\}$$

form an open cover of S_1 . Since S_1 is compact, by the Heine-Borel Theorem, there is finite number n such that

$$S_1 \subset \bigcup_{j=1}^{n} G_{k_j}\,.$$

But this means that

$$S_1 \cap S_{k_1} \cap S_{k_2} \cap \cdots \cap S_{k_n} = \emptyset\,.$$

This contradict to our nested property assumption $(S_1 \supset S_2 \supset S_3 \cdots)$.