2(p.38). In order to see that the unit sphere $S^3 = \{(x, y, z) : x^2 + y^2 + z^2 = 1\}$ is arcwise connected, we have to show that, for any $x, y \in S^3$, there is a continuous function $f : [0, 1] \to \mathbb{R}^3$ such that $f(0) = x, f(1) = y$, and $f(t) \in S^3$ for all $t \in [0, 1]$. For any two points $x, y \in S^3$ and $x \neq -y$, let us define the function

$$f(t) = \frac{(1 - t)x + ty}{|(1 - t)x + ty|}.$$  

Since $x \neq -y$, $|(1 - t)x + ty|$ can not be zero for all $t \in [0, 1]$. Thus $f(t)$ is well defined, furthermore, $f(t)$ is a continuous function and

$$f(0) = \frac{x}{|x|} = x, \quad f(1) = \frac{y}{|y|} = y \quad \text{and} \quad f(t) \in S^3 \quad \text{for all} \quad t \in [0, 1].$$

For the case $x = -y$, we pick any point $z \in S^3$ away from $x$ and $y$. Then one can define the continuous function $h(t)$ by

$$h(t) = \begin{cases} f_1(2t) & \text{if } 0 \leq t \leq 1/2, \\ f_2(2t - 1) & \text{if } 1/2 \leq t \leq 1, \end{cases}$$

where

$$f_1(t) = \frac{(1 - t)x + tz}{|(1 - t)x + tz|} \quad \text{and} \quad f_2(t) = \frac{(1 - t)z + ty}{|(1 - t)z + ty|}.$$  

Then $h(t)$ is also a continuous function and satisfies (1). Similarly one can generalize this arguments in $\mathbb{R}^n$.

4(p.38). Suppose $S_1$ and $S_2$ are connected sets in $\mathbb{R}^n$ that contain at least one point in common. Assume that $S_1 \cup S_2$ is disconnected, i.e. $S_1 \cup S_2$ is the union of two nonempty subsets $U_1$ and $U_2$, neither of which intersects the closure of the other one. Since there is at least one point in common $S_1$ and $S_2$, consider $x_0 \in S_1 \cap S_2$; $x_0$ belongs to one of subsets $U_1$ and $U_2$, say $x_0 \in U_1$. Since $S_1 \subseteq U_1 \cup U_2$, $S_1 \cap U_1 \neq \emptyset$ and $S_1$ is connected, we must have $S_1 \cap U_2 = \emptyset$. Similarly $S_2 \cap U_1 = \emptyset$. But then $(S_1 \cup S_2) \cap U_2 = \emptyset$, and it is a contradiction. In $\mathbb{R}$, $S_1 \cap S_2$ must be connected. However, in $\mathbb{R}^n (n > 1)$ it is not always true. For example, consider

$$S_1 = \{(x_1, x_2) \in \mathbb{R}^2 : x_1^2 + x_2^2 = 1 \quad \text{and} \quad x_1 \geq 0\},$$

$$S_2 = \{(x_1, x_2) \in \mathbb{R}^2 : x_1^2 + x_2^2 = 1 \quad \text{and} \quad x_1 \leq 0\}.$$  

1(p.40). Let $f : S \to \mathbb{R}^m$ is Hölder continuous function; for some $C > 0$ and $\lambda > 0$, $f$ satisfies

$$|f(x) - f(y)| \leq C|x - y|^{\lambda}$$

for all $x, y \in S$. For any given $\epsilon > 0$, choose $\delta = (\epsilon/(C + 1))^{1/\lambda}$. If $x, y \in S$ are given points satisfying

$$|x - y| < \delta$$

then

$$|f(x) - f(y)| \leq C|x - y|^{\lambda} \leq C\delta^{\lambda} \leq C\frac{\epsilon}{C + 1} < \epsilon.$$  

Thus, $f$ is uniformly continuous on $S$.

4(p. 41). Suppose that is $f : S \to \mathbb{R}^m$ is uniformly continuous on $S$ and $\{x_k\}$ is a Cauchy sequence in $S$. Since $f$ is uniformly continuous on $S$, there exist $\delta > 0$ so that

$$|f(x) - f(y)| < \epsilon \quad \text{whenever} \quad x, y \in S \quad \text{and} \quad |x - y| < \delta.$$
Since \( \{x_k\} \) is a Cauchy sequence, there exists \( K \) so that
\[
|x_k - x_j| < \delta \text{ if } k, j > K.
\]
From (2) we see that
\[
k, j > K \text{ implies } |f(x_k) - f(x_j)| < \epsilon.
\]
This prove that \( \{f(x_k)\} \) is also a Cauchy sequence.

Let \( x_k = \frac{1}{k} \) for \( k \in \mathbb{N} \). Then \( \{x_k\} \) is obviously a Cauchy sequence. We now consider the function
\[
f(x) = \frac{1}{x^2}, \quad x > 0,
\]
which is a continuous (but not uniformly continuous) function. Since
\[
f(x_k) = \frac{1}{(x_k)^2} = k^2,
\]
\( \{f(x_k)\} \) is not a Cauchy sequence.