Answer keys for the 4th week homework

6(p.62). In order to find the df_j, we calculate $\partial f_j/\partial x_i$, $j \neq i$ as follows.

$$\frac{\partial f_j}{\partial x_i} = \frac{\partial}{\partial x_i}\left(\frac{x_j}{|\mathbf{x}|}\right) = \frac{\partial}{\partial x_i}\left(\frac{x_j}{(x_i^2 + \sum_{k\neq i}^n x_k^2)^{1/2}}\right) = \frac{-x_j x_i}{(x_i^2 + \sum_{k\neq i}^n x_k^2)^{3/2}} = -\frac{x_i x_j}{|\mathbf{x}|^3}\,.$$

We also can find that

$$\frac{\partial f_j}{\partial x_j} = \frac{\partial}{\partial x_j}\left(\frac{x_j}{(x_j^2 + \sum_{k\neq j} x_k^2)^{1/2}}\right) = \frac{(x_j^2 + \sum_{k\neq j} x_k^2)^{1/2} - x_j \frac{1}{2}(x_j^2 + \sum_{k\neq j} x_k^2)^{-1/2} 2x_j}{x_j^2 + \sum_{k\neq j} x_k^2} = \frac{1}{|\mathbf{x}|} - \frac{x_j^2}{|\mathbf{x}|^3}\,.$$

Then we have

$$df_j = \sum_{i=1}^n \frac{\partial f_j}{\partial x_i} dx_i = \left(\frac{|\mathbf{x}|^2 - x_j^2}{|\mathbf{x}|^3}\right) dx_j - \sum_{i\neq j} \frac{x_i x_j dx_i}{|\mathbf{x}|^3} = \frac{1}{|\mathbf{x}|} dx_j - \sum_{i=1}^n \frac{x_i x_j dx_i}{|\mathbf{x}|^3}\,,$$

and

$$x_j df_j = \frac{x_j}{|\mathbf{x}|} dx_j - \sum_{i=1}^n \frac{x_i x_j^2 dx_i}{|\mathbf{x}|^3}\,.$$

We can evaluate that

$$\begin{aligned}
\sum_{j=1}^n x_j df_j &= \sum_{j=1}^n \frac{x_j dx_j}{|\mathbf{x}|} - \sum_{i=1}^n \frac{x_i x_j^2 dx_i}{|\mathbf{x}|^3} = \sum_{j=1}^n \frac{x_j dx_j}{|\mathbf{x}|} - \sum_{j=1}^n \sum_{i=1}^n \frac{x_i x_j^2 dx_i}{|\mathbf{x}|^3} \\
&= \sum_{j=1}^n \frac{x_j dx_j}{|\mathbf{x}|} - \sum_{i=1}^n \sum_{j=1}^n \frac{x_i x_j^2 dx_i}{|\mathbf{x}|^3} = \sum_{j=1}^n \frac{x_j dx_j}{|\mathbf{x}|} - \sum_{i=1}^n \frac{x_i dx_i}{|\mathbf{x}|^3}\left(\sum_{j=1}^n x_j^2\right) \\
&= \sum_{j=1}^n \frac{x_j dx_j}{|\mathbf{x}|} - \sum_{i=1}^n \frac{x_i dx_i}{|\mathbf{x}|} = 0\,.
\end{aligned}$$

7(p.62). (a) Choose $\delta = 2\epsilon$, and assume that $|x| < \delta$ and $|y| < \delta$. Since $f(0,0) = 0$ and $|xy| \leq \frac{1}{2}(x^2 + y^2)$, we have

$$|f(x,y) - f(0,0)| = |f(x,y)| = \left|\frac{x^2 y}{x^2 + y^2}\right| \leq \frac{|x^2 y|}{2|xy|} = \frac{|x|}{2} \leq \frac{\delta}{2} = \epsilon\,.$$

This proves the continuity of $f(x,y)$ at $(0,0)$.
(b) Let us write any unit vector in $\mathbb{R}^2$ by $\mathbf{u} = (\cos\theta, \sin\theta)$. Then, the directional derivative of f at $(0,0)$ in the direction $\mathbf{u}$ is defined by

$$\partial_{\mathbf{u}} f(0,0) = \lim_{t\to 0} \frac{f(t\cos\theta, t\sin\theta) - f(0,0)}{t}\,.$$

In fact, we can evaluate

$$\partial_{\mathbf{u}} f(0,0) = \lim_{t\to 0} \frac{1}{t} \cdot \frac{t^2\cos^2\theta + t\sin\theta}{t^2(\cos^2\theta + \sin^2\theta)} = \lim_{t\to 0} \frac{t^3\cos^2\theta\sin\theta}{t^3} = \cos^2\theta\sin\theta\,.$$

The limit always exist in any direction and we have $\partial_{(\cos\theta,\sin\theta)} f(0,0) = \cos^2\theta\sin\theta$.
(c) If f is differentiable at $(0,0)$, then the directional derivatives of f at $(0,0)$ all exist, and they must be given by

$$\partial_{(\cos\theta,\sin\theta)} f(0,0) = \nabla f(0,0) \cdot (\cos\theta, \sin\theta)\,.$$

From (b), we know $\partial_{(\cos\theta,\sin\theta)} f(0,0)$ are always exist and given by $\cos^2\theta\sin\theta$. Since

$$\nabla f(x,y) = \left(\frac{2xy(x^2+y^2) - x^2y(2x)}{(x^2+y^2)^2}, \frac{y(x^2+y^2) - x^2y(2y)}{(x^2+y^2)^2}\right),$$

$\nabla f(0,0)$ is not even exist. Thus, f is not differentiable at $(0,0)$.

4(p70.). By Chain rule,

$$\sum_{j=1}^{n}\left(\frac{\partial u}{\partial x_j}\right)^2 = \sum_{j=1}^{n}\left(\frac{\partial u}{\partial r}\frac{\partial r}{\partial x_j}\right)^2 = \left(\frac{\partial u}{\partial r}\right)^2\sum_{j=1}^{n}\left(\frac{\partial r}{\partial x_j}\right)^2 = \big(f'(r)\big)^2\sum_{j=1}^{n}\left(\frac{\partial r}{\partial x_j}\right)^2.$$

Thus, it is enough to show that $\sum_{j=1}^{n}(\partial r/\partial x_j)^2 = 1$. It is easy to see that

$$\frac{\partial r}{\partial x_j} = \frac{1}{2}(x_1^2+\cdots+x_n^2)^{-1/2}2x_j = \frac{x_j}{|\mathbf{x}|}.$$

Thus we have

$$\sum_{j=1}^{n}\left(\frac{\partial r}{\partial x_j}\right)^2 = \frac{1}{|\mathbf{x}|^2}\sum_{j=1}^{n}x_j^2 = \frac{|\mathbf{x}|^2}{|\mathbf{x}|^2} = 1.$$

1(p.72). (a) *If f is differentiable on a set containing the line segment from $\mathbf{a}$ to $\mathbf{b}$, and $f(\mathbf{a}) = f(\mathbf{b})$, then there is a point $\mathbf{c}$ on the line segment from $\mathbf{a}$ to $\mathbf{b}$ such that $\nabla f(\mathbf{c})$ is perpendicular to the line segment from $\mathbf{a}$ to $\mathbf{b}$.*

Proof. If $\mathbf{a} = \mathbf{b}$ then there is nothing to prove. We now assume that $\mathbf{a} \neq \mathbf{b}$. By the mean value theorem, we know that there is a point $\mathbf{c}$ on the line segment from $\mathbf{a}$ to $\mathbf{b}$ so that

$$f(\mathbf{b}) - f(\mathbf{a}) = \nabla f(\mathbf{c})\cdot(\mathbf{b}-\mathbf{a}).$$

Since $f(\mathbf{a}) = f(\mathbf{b})$ and $\mathbf{a} \neq \mathbf{b}$, there must be a point $\mathbf{c}$ on the line segment from $\mathbf{a}$ to $\mathbf{b}$ such that $\nabla f(\mathbf{c})$ is perpendicular to the line segment from $\mathbf{a}$ to $\mathbf{b}$.