Answer keys for the 5th week homework

2(p.76). (a) Solving the equations explicitly for y and z returns us

$$x^2 - y^2 = 2x + 4y \Rightarrow y^2 + 4y + 4 = x^2 - 2x + 4 \Rightarrow (y+2)^2 = x^2 - 2x + 4$$

$$\Rightarrow y + 2 = \pm\sqrt{x^2 - 2x + 4} \Rightarrow y = \pm\sqrt{x^2 - 2x + 4} - 2\,, \tag{1}$$

and

$$z = x^2 - \left(\frac{z - 2x}{4}\right)^2 \Rightarrow z^2 - 4(x-4)z = 12x^2 \Rightarrow \big(z - 2(x-4)\big)^2 = 16x^2 - 32x + 16$$

$$\Rightarrow z - 2(x-4) = \pm\sqrt{16x^2 - 32x + 16} \Rightarrow z = \pm\sqrt{16x^2 - 32x + 16} + 2(x-4)\,. \tag{2}$$

Thus we can find

$$\begin{aligned}
\frac{dy}{dx} &= \pm\frac{1}{2}(x^2 - 2x + 4)^{-1/2}(2x - 2) = \pm\frac{x - 1}{\sqrt{x^2 - 2x + 4}}\,,\\
\frac{dz}{dx} &= \pm\frac{1}{2}(16x^2 - 32x + 16)^{-1/2}(32x - 32) + 2 = \pm\frac{16(x-1)}{\sqrt{16x^2 - 32x + 16}} + 2\,.
\end{aligned}$$

Using the first equations in (1) and (2), we have

$$\frac{dy}{dx} = \frac{x-1}{y+2}\,,\ y \neq -2 \text{ and } \frac{dz}{dx} = \frac{16(x-1)}{z - 2(x-4)} + 2 = \frac{12x + 2z}{z - 2x + 8}\,,\ z \neq 2x - 8 (\text{or } y \neq -2)\,.$$

(b) In order to find the dy/dx and dz/dx, we take the derivative, d/dx, for both given equation;

$$\frac{dz}{dx} = 2x - 2y\frac{dy}{dx}, \quad \frac{dz}{dx} = 2 + 4\frac{dy}{dx}\,.$$

Thus, we get

$$2x - 2y\frac{dy}{dx} = 2 + 4\frac{dy}{dx} \Rightarrow \frac{dy}{dx} = \frac{x-1}{y+2}\,,\ y \neq -2$$

and

$$\begin{aligned}
\frac{dz}{dx} &= 2x - \frac{y}{2}\left(\frac{dz}{dx} - 2\right) \Rightarrow (y+2)\frac{dz}{dx} = 4x + 2y\\
\Rightarrow \frac{dz}{dx} &= \frac{4x + 2y}{y + 2} = \frac{4x + 2\left(\frac{z-2x}{4}\right)}{\frac{z-2x}{4} + 2} = \frac{12x + 2z}{z - 2x + 8}\,,\ z \neq 2x - 8 (\text{or } y \neq -2)\,.
\end{aligned}$$

6(p.77). By the assumption, we can write

$$x = f(y,z),\ y = g(x,z),\ \text{and } z = k(x,y)\,.$$

Thus we have

$$F(f(y,z), y, z) = 0,\ F(x, g(x,z), z) = 0,\ \text{and } F(x, y, k(x,y)) = 0\,.$$

By the formula (2.44) P.74, we have

$$\frac{\partial x}{\partial y} = \frac{\partial f}{\partial y} = -\frac{F_y}{F_x},\ \frac{\partial y}{\partial z} = \frac{\partial g}{\partial z} = -\frac{F_z}{F_y},\ \text{and } \frac{\partial z}{\partial x} = \frac{\partial k}{\partial x} = -\frac{F_x}{F_z}\,.$$

Thus

$$\frac{\partial x}{\partial y}\frac{\partial y}{\partial z}\frac{\partial z}{\partial x} = \left(-\frac{F_y}{F_x}\right)\left(-\frac{F_z}{F_y}\right)\left(-\frac{F_x}{F_z}\right) = -\frac{F_yF_zF_x}{F_xF_yF_z} = -1\,.$$

5(p.84). As in the proof of the Euler's Theorem, we consider the function $\varphi(t) = f(t\mathbf{x})$, then we have $\varphi''(t) = a(a-1)t^{a-2}f(\mathbf{x}) = a(a-1)t^{-2}f(t\mathbf{x})$. On the other hand, by the chain rule, we have

$$\begin{aligned}
\varphi''(t) = \frac{d}{dt}\Big[\mathbf{x}\cdot\nabla f(t\mathbf{x})\Big] &= \frac{d}{dt}\Big[\sum_{j=1}^n x_j\partial_j f(t\mathbf{x})\Big] \\
&= \sum_{j=1}^n x_j\Big(\nabla\partial_j f(t\mathbf{x})\cdot\frac{d}{dt}(t\mathbf{x})\Big) = \sum_{j=1}^n x_j\Big(\mathbf{x}\cdot\nabla\partial_j f(t\mathbf{x})\Big) \\
&= \sum_{j=1}^n x_j\Big(\sum_{k=1}^n x_k\partial_k\partial_j f(t\mathbf{x})\Big) = \sum_{j=1}^n\sum_{k=1}^n x_jx_k\partial_j\partial_k f(t\mathbf{x})\,.
\end{aligned}$$

Setting $t = 1$ yields the desire result.

10(p.95). By the equation (2.70) and the assumption, we have

$$\begin{aligned}
|R_{\mathbf{a},k}(\mathbf{h})| &= \Big|k\sum_{|\alpha|=k}\frac{\mathbf{h}^\alpha}{\alpha!}\int_0^1(1-t)^{k-1}[\partial^\alpha f(\mathbf{a}+t\mathbf{h}) - \partial^\alpha f(\mathbf{a})]\,dt\Big| \\
&\le k\sum_{|\alpha|=k}\frac{|\mathbf{h}|^\alpha}{\alpha!}\int_0^1(1-t)^{k-1}\big|\partial^\alpha f(\mathbf{a}+t\mathbf{h}) - \partial^\alpha f(\mathbf{a})\big|\,dt \\
&\le k\sum_{|\alpha|=k}\frac{|\mathbf{h}|^\alpha}{\alpha!}\int_0^1(1-t)^{k-1}C|\mathbf{a}+t\mathbf{h}-\mathbf{a}|^\lambda\,dt \\
&= Ck|\mathbf{h}|^\lambda\sum_{|\alpha|=k}\frac{|\mathbf{h}|^\alpha}{\alpha!}\int_0^1(1-t)^{k-1}t^\lambda\,dt
\end{aligned}$$

Since $0 \le t \le 1$, we can estimate the integral as follows

$$\int_0^1(1-t)^{k-1}t^\lambda\,dt \le \int_0^1(1-t)^{k-1}\,dt = \frac{1}{k}\,,$$

and we have

$$|R_{\mathbf{a},k}(\mathbf{h})| \le C|\mathbf{h}|^\lambda\sum_{|\alpha|=k}\frac{|\mathbf{h}|^\alpha}{\alpha!} = C'|\mathbf{h}|^{k+\lambda}\,,$$

where the last equality is due to the multinomial theorem.