Answer keys for the 7th week homework

5(p.111). Let us denote an $m \times n$ matrix $A = (A_{jk})$, $\mathbf{x} = (x_1, x_2, \cdots, x_n)$ and $b = (b_1, b_2, \cdots, b_m)$. Then we can write the function

$$\mathbf{f}(\mathbf{x}) = \begin{pmatrix} f_1(\mathbf{x}) \\ \vdots \\ f_m(\mathbf{x}) \end{pmatrix} = \begin{pmatrix} \sum_{k=1}^n A_{1k}x_k + b_1 \\ \vdots \\ \sum_{k=1}^n A_{mk}x_k + b_m \end{pmatrix} .$$

Since, for all $1 \le j \le m$, $\nabla f_j(\mathbf{x}) = \nabla\Big(\sum_{k=1}^n A_{jk}x_k + b_j \Big) = (A_{j1}, A_{j2}, \cdots, A_{jn})$, we have

$$D\mathbf{f} = \begin{pmatrix} \frac{\partial f_1}{\partial x_1} & \cdots & \frac{\partial f_1}{\partial x_n} \\ \vdots & & \vdots \\ \frac{\partial f_m}{\partial x_1} & \cdots & \frac{\partial f_m}{\partial x_n} \end{pmatrix} = \begin{pmatrix} A_{11} & \cdots & A_{1n} \\ \vdots & & \vdots \\ A_{m1} & \cdots & A_{mn} \end{pmatrix} = A .$$

7(p.111). Since $\mathbf{f}(\mathbf{x}) = (f_1(\mathbf{x}), \cdots, f_m(\mathbf{x}))$, $\mathbf{g}(\mathbf{x}) = (g_1(\mathbf{x}), \cdots, g_m(\mathbf{x}))$ and $\mathbf{x} = (x_1, \cdots, x_n)$, we can have a real-valued function on $\mathbb{R}^n$

$$h(\mathbf{x}) = \mathbf{f}(\mathbf{x}) \cdot \mathbf{g}(\mathbf{x}) = \sum_{j=1}^m f_j(\mathbf{x}) g_j(\mathbf{x}) .$$

Thus

$$\begin{aligned} \nabla h(\mathbf{x}) &= \begin{pmatrix} \frac{\partial h(\mathbf{x})}{\partial x_1} \\ \vdots \\ \frac{\partial h(\mathbf{x})}{\partial x_n} \end{pmatrix} = \begin{pmatrix} \frac{\partial}{\partial x_1} \sum_{j=1}^m f_j(\mathbf{x}) g_j(\mathbf{x}) \\ \vdots \\ \frac{\partial}{\partial x_n} \sum_{j=1}^m f_j(\mathbf{x}) g_j(\mathbf{x}) \end{pmatrix} = \begin{pmatrix} \sum_{j=1}^m \frac{\partial f_j(\mathbf{x})}{\partial x_1} g_j(\mathbf{x}) + f_j(\mathbf{x}) \frac{\partial g_j(\mathbf{x})}{\partial x_1} \\ \vdots \\ \sum_{j=1}^m \frac{\partial f_j(\mathbf{x})}{\partial x_n} g_j(\mathbf{x}) + f_j(\mathbf{x}) \frac{\partial g_j(\mathbf{x})}{\partial x_n} \end{pmatrix} \\ &= \begin{pmatrix} \sum_{j=1}^m \frac{\partial f_j(\mathbf{x})}{\partial x_1} g_j(\mathbf{x}) \\ \vdots \\ \sum_{j=1}^m \frac{\partial f_j(\mathbf{x})}{\partial x_n} g_j(\mathbf{x}) \end{pmatrix} + \begin{pmatrix} \sum_{j=1}^m f_j(\mathbf{x}) \frac{\partial g_j(\mathbf{x})}{\partial x_1} \\ \vdots \\ \sum_{j=1}^m f_j(\mathbf{x}) \frac{\partial g_j(\mathbf{x})}{\partial x_n} \end{pmatrix} \\ &= \begin{pmatrix} \frac{\partial f_1(\mathbf{x})}{\partial x_1} & \cdots & \frac{\partial f_m(\mathbf{x})}{\partial x_1} \\ \vdots & & \vdots \\ \frac{\partial f_1(\mathbf{x})}{\partial x_n} & \cdots & \frac{\partial f_m(\mathbf{x})}{\partial x_n} \end{pmatrix} \begin{pmatrix} g_1(\mathbf{x}) \\ \vdots \\ g_m(\mathbf{x}) \end{pmatrix} + \begin{pmatrix} \frac{\partial g_1(\mathbf{x})}{\partial x_1} & \cdots & \frac{\partial g_m(\mathbf{x})}{\partial x_1} \\ \vdots & & \vdots \\ \frac{\partial g_1(\mathbf{x})}{\partial x_n} & \cdots & \frac{\partial g_m(\mathbf{x})}{\partial x_n} \end{pmatrix} \begin{pmatrix} f_1(\mathbf{x}) \\ \vdots \\ f_m(\mathbf{x}) \end{pmatrix} \\ &= [D\mathbf{f}(\mathbf{x})]^* \mathbf{g}(\mathbf{x}) + [D\mathbf{g}(\mathbf{x})]^* \mathbf{f}(\mathbf{x}) . \end{aligned}$$

1(p.119). Let $F(x, y, z) = x^2 - 4x + 2y^2 - yz - 1$. Then $F(2, -1, 3) = 0$,

$$\nabla F(x, y, z) = (2x - 4, 4y - z, -y)$$

and $\nabla F(2, 1, 3) = (0, -7, 1)$. Thus, by the implicit function theorem, we know that there is a neighborhood N of $(2, 1, 3)$ so that the equation $F(x, y, z) = 0$ can be solved uniquely for y and z , but $F(x, y, z) = 0$ cannot be solved uniquely for x as a function of y and z in any neighborhood of $(2, 1, 3)$. Explicitly, one can solve $F = 0$ for x, $x = 2 \pm \sqrt{-2y^2 + yz + 5}$. But $-2y^2 + yz + 5 < 0$ for any $z > 3$ and $y = 1$. This coincides with the implicit function theorem result. One also solve

$F = 0$, for y,

$$y = \frac{z}{4} + \sqrt{\frac{1}{2}\left(1 - x^2 + 4x + \frac{z^2}{8}\right)}, \quad y > z/4$$
$$y = \frac{z}{4} - \sqrt{\frac{1}{2}\left(1 - x^2 + 4x + \frac{z^2}{8}\right)}, \quad y < z/4\,,$$

and for any $2 - \sqrt{5} \le x \le 2 + \sqrt{5}$ and $z \in \mathbb{R}$, $1 - x^2 + 4x + \frac{z^2}{8} \ge 0$. Finally, for z, we have

$$z = \frac{x^2 - 4x + 2y^2 - 1}{y}, \quad y \neq 0\,.$$

8(p.120). Let $F = xy^2 + xzu + yv^2 - 3$ and $G = u^3yz + 2xv - u^2v^2 - 2$. Then

$$\frac{\partial(F,G)}{\partial(u,v)} = det\begin{pmatrix} xz & 2vy \\ 3u^2yz - 2uv^2 & 2x - 2u^2v \end{pmatrix} = 2xz(x - u^2v) - 2vy(3u^2yz - 2uv^2)$$

and

$$\left.\frac{\partial(F,G)}{\partial(u,v)}\right|_{(1,1,1,1)} = -2 \neq 0\,.$$

Thus, one can solve the system of equations $F = 0$ and $G = 0$ for u and v as functions of x, y, and z near $x = y = z = u = v = 1$.