Answer keys for the 7th week homework

5(p.111). Let us denote an $m \times n$ matrix $A = (A_{jk})$, $x = (x_1, x_2, \ldots, x_n)$ and $b = (b_1, b_2, \ldots, b_m)$. Then we can write the function

$$f(x) = \begin{pmatrix} f_1(x) \\ \vdots \\ f_m(x) \end{pmatrix} = \begin{pmatrix} \sum_{k=1}^n A_{1k}x_k + b_1 \\ \vdots \\ \sum_{k=1}^n A_{mk}x_k + b_m \end{pmatrix}.$$ 

Since, for all $1 \leq j \leq m$, $\nabla f_j(x) = \nabla \left( \sum_{k=1}^n A_{jk}x_k + b_j \right) = (A_{j1}, A_{j2}, \ldots, A_{jn})$, we have

$$Df = \begin{pmatrix} \frac{\partial f_1}{\partial x_1} & \cdots & \frac{\partial f_1}{\partial x_n} \\ \vdots & \ddots & \vdots \\ \frac{\partial f_m}{\partial x_1} & \cdots & \frac{\partial f_m}{\partial x_n} \end{pmatrix} = \begin{pmatrix} A_{11} & \cdots & A_{1n} \\ \vdots & \ddots & \vdots \\ A_{m1} & \cdots & A_{mn} \end{pmatrix} = A.$$

7(p.111). Since $f(x) = (f_1(x), \ldots, f_m(x))$, $g(x) = (g_1(x), \ldots, g_m(x))$ and $x = (x_1, \ldots, x_n)$, we can have a real-valued function on $\mathbb{R}^n$

$$h(x) = f(x) \cdot g(x) = \sum_{j=1}^m f_j(x)g_j(x).$$

Thus

$$\nabla h(x) = \begin{pmatrix} \frac{\partial h(x)}{\partial x_1} \\ \vdots \\ \frac{\partial h(x)}{\partial x_n} \end{pmatrix} = \begin{pmatrix} \frac{\partial}{\partial x_1} \sum_{j=1}^m f_j(x)g_j(x) \\ \vdots \\ \frac{\partial}{\partial x_n} \sum_{j=1}^m f_j(x)g_j(x) \end{pmatrix} = \begin{pmatrix} \sum_{j=1}^m \frac{\partial f_j(x)}{\partial x_1}g_j(x) + f_j(x)\frac{\partial g_j(x)}{\partial x_1} \\ \vdots \\ \sum_{j=1}^m \frac{\partial f_j(x)}{\partial x_n}g_j(x) + f_j(x)\frac{\partial g_j(x)}{\partial x_n} \end{pmatrix} = \sum_{j=1}^m \frac{\partial f_j(x)}{\partial x_j}g_j(x) + f_j(x)\frac{\partial g_j(x)}{\partial x_j} = \left[ Df(x) \right]^* g(x) + [Dg(x)]^* f(x).$$

1(p.119). Let $F(x, y, z) = x^2 - 4x + 2y^2 - yz - 1$. Then $F(2, -1, 3) = 0$,

$$\nabla F(x, y, z) = (2x - 4, 4y - z, -y)$$

and $\nabla F(2, 1, 3) = (0, -7, 1)$. Thus, by the implicit function theorem, we know that there is a neighborhood $N$ of $(2,1,3)$ so that the equation $F(x, y, z) = 0$ can be solved uniquely for $y$ and $z$, but $F(x, y, z) = 0$ cannot be solved uniquely for $x$ as a function of $y$ and $z$ in any neighborhood of $(2, 1, 3)$. Explicitly, one can solve $F = 0$ for $x$, $x = 2 ± \sqrt{-2y^2 + yz + 5}$. But $-2y^2 + yz + 5 < 0$ for any $z > 3$ and $y = 1$. This coincides with the implicit function theorem result. One also solve
\( F = 0 \), for \( y \),

\[
y = \frac{z}{4} + \sqrt{\frac{1}{2}\left(1 - x^2 + 4x + \frac{z^2}{8}\right)}, \quad y > z/4
\]

\[
y = \frac{z}{4} - \sqrt{\frac{1}{2}\left(1 - x^2 + 4x + \frac{z^2}{8}\right)}, \quad y < z/4,
\]

and for any \( 2 - \sqrt{5} \leq x \leq 2 + \sqrt{5} \) and \( z \in \mathbb{R}, 1 - x^2 + 4x + \frac{z^2}{8} \geq 0 \). Finally, for \( z \), we have

\[
z = \frac{x^2 - 4x + 2y^2 - 1}{y}, \quad y \neq 0.
\]

8(p.120). Let \( F = xy^2 + xzu + yv^2 - 3 \) and \( G = u^3yz + 2xv - u^2v^2 - 2 \). Then

\[
\frac{\partial (F, G)}{\partial (u, v)} = \det \begin{pmatrix} xz & 2vy \\ 3u^2yz - 2uv^2 & 2x - 2u^2v \end{pmatrix} = 2xz(x - u^2v) - 2vy(3u^2yz - 2uv^2)
\]

and

\[
\frac{\partial (F, G)}{\partial (u, v)} \bigg|_{(1,1,1,1)} = -2 \neq 0.
\]

Thus, one can solve the system of equations \( F = 0 \) and \( G = 0 \) for \( u \) and \( v \) as functions of \( x, y, \) and \( z \) near \( x = y = z = u = v = 1 \).