THE $cl$-core OF AN IDEAL

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Abstract. We expand the notion of core to $cl$-core for Nakayama closures $cl$. In the characteristic $p > 0$ setting, when $cl$ is the tight closure, we give some examples of ideals when the core and the $*$-core differ. Moreover, we show that the $*$-core$(I) = \text{core}(I)$, if $I$ is an ideal in a one-dimensional domain with infinite residue field or if $I$ is an ideal generated by a system of parameters in any Noetherian ring. More generally, we show the same result in a local Cohen–Macaulay normal domain with perfect infinite residue field, if the analytic spread, $\ell$, is equal to the $*$-spread and $I$ is $G_\ell$ and weakly-$(\ell - 1)$-residually $S_2$. This last is dependent on our result that generalizes the notion of general reductions to general minimal $*$-reductions. We also determine that the $*$-core of a tightly closed ideal in certain one-dimensional semigroup rings is tightly closed and therefore integrally closed.

1. Introduction

The core of an ideal, the intersection of all reductions of the ideal, was introduced by Rees and Sally in [RS] in the 80’s. Then over a decade past before Huneke and Swanson [HS1] analyzed the core of ideals in 2-dimensional regular local rings. Then a stream of papers came out within a decade by Corso, Polini and Ulrich [CPU1], [CPU2], [PU], Hyry and Smith [HyS1], [HyS2] and Huneke and Trung [HT] expanding the understanding and computability of core. As it is the intersection of reductions, in general it lies deep within the ideal. In fact, the core is related to the Briançon-Skoda Theorem [LS]: Let $(R, \mathfrak{m})$ be a regular local ring, then $\overline{I \mathfrak{m}^d} \subseteq J$ for any reduction $J$ of $I$. Hence, $\overline{I \mathfrak{m}^d} \subseteq \text{core}(I)$. A very slick proof of the Briançon-Skoda Theorem was given in characteristic $p > 0$, using tight closure, [HH, Theorem 5.4]. We would like to expand the notion of core to other closure operations; in particular, Nakayama closure operations. Epstein defined the notion of Nakayama closure as follows:

Definition 1.1. ([Ep]) A closure operation $cl$, defined on a Noetherian local ring $(R, \mathfrak{m})$ is a Nakayama closure if for all ideals $J \subset I \subset (J + \mathfrak{m}I)^d$, then $I \subset J^d$.

Note that integral closure, tight closure and Frobenius closure are examples of Nakayama closures, [Ep, Proposition 2.1]. Recall, that both the tight closure and the Frobenius closure are characteristic $p > 0$ notions. It may be that we can formulate

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some Briançon-Skoda like Theorems for other cores such as tight closure using our new definition of $cl$-core.

Epstein’s main reason for the definition of Nakayama closure was to expand the notion of reduction and spread to these other closure operations. With a well defined notion of reduction and spread, we can easily extend the notion of core to these other closure operations. In general, the $cl$-cores will not lie as deep in the ideal as the core itself. This will follow from the fact that the partial ordering of closure operations leads to a reverse partial ordering on the $cl$-cores. Our hope in studying these $cl$-cores is that tight closure methods may be used to compute the core in situations where the core and the $*$-core agree.

In Section 2, we provide some background information about the core and tight closure theory, along with a review of some central theorems that are used in this article. In Section 3, we review $cl$-reductions of ideals. We also discuss the $cl$-spread of an ideal and define both the $cl$-deviation and the second $cl$-deviation in terms of the $cl$-spread. We also introduce the notion of $cl$-core. In Section 4, we show different instances when the core and the $*$-core agree. Our main result, Theorem 4.5, shows that we can form general $*$-reductions. This allows us to show in particular that if $(R, m)$ is Gorenstein normal domain of positive characteristic with infinite perfect field with the test ideal equal to $m$ and $I$ is an $m$-primary tightly closed ideal then $*\text{-core}(I) = \text{core}(I)$. Also, when $(R, m)$ is a Cohen–Macaulay normal domain with perfect infinite residue field of prime characteristic and $I$ is an ideal which is $G_\ell$ and weakly residually $(\ell - 1)\cdot S_2$ with $\ell^*(I) = \ell(I) = \ell$ then $\text{core}(I) = *\text{-core}(I)$.

In Section 5, we discuss when the $*$-core is tightly closed in some one-dimensional semigroup rings. In Section 6, we give some examples of the $*$-core of an ideal and in each case we compare the core with the $*$-core.

### 2. Background

In this section we recall some notions that we will use extensively in this article and also recall some results that we use.

**Definition 2.1.** Let $R$ be a Noetherian local ring of prime characteristic $p > 0$. We denote positive powers of $p$ by $q$ and the set of elements of $R$ which are not contained in the union of minimal primes by $R^a$. Then

(a) For any ideal $I \subset R$, $I^{[q]}$ is the ideal generated by the $q$th powers of elements in $I$.

(b) We say an element $x \in R$ is in the tight closure, $I^*$, of $I$ if there exists a $c \in R^a$, such that $cx^q \in I^{[q]}$ for all large $q$.

(c) We say an element $x \in R$ is in the Frobenius closure $I^F$ of $I$ if $x^q \in I^{[q]}$ for all large $q$.

Finding the tight closure of an ideal would be hard without test elements and test ideals. A test element is an element $c \in R$ which is not in any minimal prime and
$cI^* \subset I$ for all $I \subset R$. Note that $c \in \bigcap_{I \subset R} (I : I^*)$. Since the intersection of ideals is an ideal we call the ideal $\tau = \bigcap_{I \subset R} (I : I^*)$ the test ideal, i.e. the ideal generated by all the test elements. We say that $I$ is a parameter ideal if $I$ is generated by part of a system of parameters. In a Gorenstein isolated singularity, the following theorem of Smith [Sm] gives a nice way to compute the tight closure of a parameter ideal using the test ideal.

**Theorem 2.2.** ([Sm, Lemma 3.6, Proposition 4.5]) Let $(R, \mathfrak{m})$ be a Gorenstein isolated singularity with $\mathfrak{m}$-primary test ideal $\tau$. Then for any system of parameters $x_1, x_2, \ldots, x_d$,

$$(x_1, x_2, \ldots, x_d) : \tau = (x_1, x_2, \ldots, x_d)^*.$$  

Related concepts are parameter test elements and parameter test ideals. A parameter test element is an element $c \in R$ which is not in any minimal prime and $cI^* \subset I$ for all parameter ideals $I \subset R$. Note that $c \in \bigcap_{I \subset R} (I : I^*)$. Let $P(R)$ be the set of parameter ideals in $R$. We call $\tau_{\text{par}} = \bigcap_{I \in P(R)} (I : I^*)$ the parameter test ideal. It is known in a Gorenstein ring that $\tau = \tau_{\text{par}}$. We can relax the Gorenstein assumption from the above theorem and we obtain:

**Theorem 2.3.** ([Va1]) Let $(R, \mathfrak{m})$ be a Cohen–Macaulay isolated singularity with $\mathfrak{m}$-primary parameter test ideal $\tau_{\text{par}}$. For any system of parameters $x_1, x_2, \ldots, x_d$,

$$(x_1, x_2, \ldots, x_d) : \tau_{\text{par}} = (x_1, x_2, \ldots, x_d)^*.$$  

Note, if the parameter test ideal is known to be $\mathfrak{m}$, even in a Cohen–Macaulay ring, the test ideal will also be $\mathfrak{m}$.

Another result that we will use repeatedly is the following due to Aberbach:

**Proposition 2.4.** ([Ab, Proposition 2.4]) Let $(R, \mathfrak{m})$ be an excellent, analytically irreducible local ring of characteristic $p$, let $I$ be an ideal, and let $f \in R$. Assume that $f \notin I^*$. Then there exists $q_0 = p^{e_0}$ such that for all $q \geq q_0$ we have $I^{[q]} : f^q \subset \mathfrak{m}^{[q]}$.  

Notice that later on we will be assuming that $R$ is an excellent normal local ring, which implies that $R$ is analytically irreducible, since the completion of an excellent normal ring is again normal, and thus a domain. Hence one may use Proposition 2.4.

Let $R$ be a Noetherian ring and $I$ an $R$-ideal. We say that $J \subset I$ is a reduction of $I$ if $I^{n+1} = JI^n$ for some nonnegative integer $n$. Northcott and Rees introduced this notion in [NR] in order to study multiplicities. The ideal $I$ and it’s reduction $J$ have the same multiplicity and thus one would want to shift the attention from $I$ to such a simpler ideal $J$. If $R$ is a Noetherian local ring with infinite residue field then $I$ has infinitely many reductions ([NR]). A reduction $J$ of $I$ is called minimal if it is minimal with respect to inclusion. To facilitate this lack of uniqueness for minimal reductions, Rees and Sally introduced the core of an ideal:
Definition 2.5. ([RS]) Let $R$ be a Noetherian local ring with infinite residue field. Let $I$ be an $R$-ideal. Then $\text{core}(I) = \bigcap_{J \subset I} J$ where $J$ is a reduction of $I$.

This could be further refined by taking the intersection over all minimal reductions. There has been a significant effort by several authors to find efficient ways of computing this infinite intersection. One result in particular is of special interest to us.

Theorem 2.6. ([CPU1, Theorem 4.5]) Let $R$ be a local Cohen–Macaulay ring with infinite residue field and $I$ an $R$-ideal of analytic spread $\ell$. Assume that $I$ is $G_\ell$ and weakly $(\ell - 1)$-residually $S_2$. Then $\text{core}(I) = a_1 \cap \ldots \cap a_t$ for $a_1, \ldots, a_t$ general $\ell$-generated ideals in $I$ which are reductions of $I$ and for some $t$ finite integer.

We now explain the conditions in the statement of Theorem 2.6.

The analytic spread of $I$, $\ell(I)$, is the Krull dimension of the special fiber ring of, $\mathcal{F}(I) := \bigoplus_{i \geq 0} I^i/m^i$, of $I$. It is well known that if $R$ is a Noetherian local ring with infinite residue field then any minimal reduction $J$ of $I$ has the same minimal number of generators, namely $\mu(J) = \ell(I)$, [NR]. It is straightforward to see that in general $\text{ht } I \leq \ell(I) \leq \dim R$.

Following the definitions given in [CEU] we say that an ideal $I$ satisfies the property $G_s$ if for every prime ideal $p$ containing $I$ with $\dim R_p \leq s - 1$, the minimal number of generators, $\mu(I_p)$, of $I_p$ is at most $\dim R_p$. A proper ideal $K$ is called an $s$-residual intersection of $I$ if there exists an $s$-generated ideal $a \subset I$ so that $K = a : I$ and height of $K$ is at least $s \geq g = \text{ht } I$. If the height of $I + K$ is at least $s + 1$, then $K$ is said to be a geometric $s$-residual intersection of $I$. If $R/K$ is Cohen–Macaulay for every $i$-residual intersection (geometric $i$-residual intersection) $K$ of $I$ and every $i \leq s$ then $I$ satisfies $AN_s$ ($AN_s^-$). An ideal $I$ is called $s$-residually $S_2$ (weakly $s$-residually $S_2$) if $R/K$ satisfies Serre’s condition $S_2$ for every $i$-residual intersection (geometric $i$-residual intersection) $K$ of $I$ and every $i \leq s$.

Remark 2.7. Let $(R, m)$ be a local Noetherian ring and $I$ an $R$-ideal. Let $g = \text{ht } I$.

The condition $G_s$ is not difficult to be satisfied. If $I$ is an $m$-primary ideal or in general an equimultiple ideal, i.e. $\ell = \ell(I) = \text{ht } I$, then $I$ satisfies $G_\ell$ automatically.

If $(R, m)$ is a local Cohen–Macaulay ring of dimension $d$ and $I$ an $R$-ideal satisfying $G_s$, then $I$ is universally $s$-residually $S_2$ in the following cases:

(a) $R$ is Gorenstein, and the local cohomology modules $H^{d-g-j}_m(R/I^j)$ vanish for all $1 \leq j \leq s - g + 1$, or equivalently, $\text{Ext}^{d+1}_R(R/I^j, R) = 0$ for all $1 \leq j \leq s - g + 1$ ([CEU, Theorem 4.1 and 4.3]).

(b) $R$ is Gorenstein, depth $R/I^j \geq \dim R/I - j + 1$ for all $1 \leq j \leq s - g + 1$ ([U, Theorem 2.9(a)]).

(c) $I$ has sliding depth ([HVV, Theorem 3.3]).
Notice that condition (b) implies (a) and the property $AN_s$ by [U, Theorem 2.9(a)]. Also the conditions (b) and (c) are satisfied by strongly Cohen–Macaulay ideals, i.e. ideals whose Koszul homology modules are Cohen–Macaulay. If $I$ is a Cohen–Macaulay almost complete intersection or a Cohen–Macaulay deviation two ideal of a Gorenstein ring [AH, p. 259] then $I$ is a strongly Cohen–Macaulay. Furthermore, if $I$ is in the linkage class of a complete intersection [Hu1, Theorem 1.11] then $I$ is again a strongly Cohen–Macaulay ideal. Standard examples include perfect ideals of height two and perfect Gorenstein ideals of height three.

3. $cl$-Reductions and the definition of $cl$-core

Recall that $J \subset I$ is a reduction of an ideal $I$ if $JI^n = I^{n+1}$. If $J$ is a reduction of $I$, then $J \subset I \subset \overline{J}$. Epstein defines a $cl$-reduction of an ideal $I$ to be an ideal $J \subset I \subset \overline{J}$. Epstein defines a $cl$-reduction of an ideal $I$ to be an ideal $J \subset I \subset \overline{J}$. If $cl$ is a Nakayama closure we have the following Lemma:

Lemma 3.1. ([Ep, Lemma 2.2]) If $cl$ is a Nakayama closure, then for any $cl$-reduction $J$ of $I$, there is a minimal $cl$-reduction $K$ of $I$ contained in $J$. Moreover, in this situation any minimal generating set of $K$ extends to a minimal generating set of $J$.

This Lemma shows in particular that minimal $cl$-reductions exist. Following the idea in Definition 2.5 we now define the $cl$-core.

Definition 3.2. Let $(R, m)$ be a Noetherian local ring and $cl$ a closure defined on $R$. The $cl$-core of an ideal $I$, $cl$-core$(I) = \bigcap_{J \subset I} J$ where $J$ is a $cl$-reduction of $I$.

Recall, an ideal is basic if it does not have any nontrivial reductions. We will say that an ideal is $cl$-basic if it does not have any nontrivial $cl$-reductions. Clearly if $I$ is a basic ideal $core(I) = I$. If $I$ is a $cl$-basic ideal then $cl$-core$(I) = I$. Note that we can restrict the intersection to the minimal $cl$-reductions of $I$. In [Va2], the second author has discussed the partial ordering on the set of closure operations of a ring defined as follows: If $cl_1$ and $cl_2$ are closure operations we say that $cl_1 \leq cl_2$ if and only if $I^{cl_1} \subset I^{cl_2}$.

Lemma 3.3. Let $cl_1$ be a closure operation and $cl_2$ be Nakayama closure operation defined on a Noetherian ring $R$ with $cl_1 \leq cl_2$. Let $I$ be an ideal. If $J_1$ is a minimal $cl_1$-reduction of $I$ then there exists a minimal $cl_2$-reduction $J_2$ of $I$ with $J_2 \subset J_1$.

Proof. Notice that $J_1 \subset I \subset J_1^{cl_1}$, as $J_1$ is a $cl_1$-reduction of $I$. Since $cl_1 \leq cl_2$ then $K^{cl_1} \subset K^{cl_2}$ for all ideals $K \subset R$. Hence $J_1^{cl_1} \subset J_1^{cl_2}$ and $J_1 \subset I \subset J_1^{cl_1} \subset J_1^{cl_2}$. So $J_1$ is a $cl_2$-reduction of $I$ also. Now by Lemma 3.1, there is a minimal reduction of $I$ contained in $J_1$. □

One consequence of Lemma 3.3 is the following:
Proposition 3.4. Let $cl_1$ be a closure operation and $cl_2$ be Nakayama closure operation defined on a Noetherian ring $R$ with $cl_1 \leq cl_2$. Let $I$ be an ideal. $cl_2$-core($I$) $\subseteq$ $cl_1$-core($I$).

Proof. We know that $cl_1$-core($I$) = $\bigcap_{J \subseteq I} J_1$ where $J_1$ is a $cl_1$-reduction of $I$. Now for every $J_1$, a $cl_1$-reduction of $I$ there exists a minimal $cl_2$-reduction, $J_2$ contained in $J_1$ by Lemma 3.3. Clearly, $cl_2$-core($I$) $\subseteq$ $\bigcap_{J_2 \subseteq J_1 \subseteq I} J_2 \subseteq \bigcap_{J_1 \subseteq I} J_1$ where $J_2$ are minimal $cl_2$-reductions of $J_1$. □

Note that $I^F \subseteq I^* \subseteq \overline{I}$. The first inclusion is clear as $x \in I^F$ if $x^q \in I^{[q]}$ for all $q >> 0$ implies that $cx^q \in I^{[q]}$ for some $c \in R^p$. The second inclusion holds, by [HH, Theorem 5.2]. In particular, we have the following corollary regarding the Frobenius or $F$-core, the $*$-core and the core, which is a $cl$-core where $cl$ is the integral closure.

Corollary 3.5. Let $R$ be an excellent analytically irreducible local domain of characteristic $p$ then core($I$) $\subseteq$ $*$-core($I$) $\subseteq$ $F$-core($I$) for all ideals $I$ in $R$.

Mimicking the following Proposition in [HS2, Proposition 17.8.9] we see:

Corollary 3.6. Let $R$ be a Noetherian local ring, then $\sqrt{I} = \sqrt{cl$-core($I$)}$ for any $cl \leq^*$. In particular, if $R$ is an excellent analytically irreducible local domain of characteristic $p$, $\sqrt{I} = \sqrt{*$-core($I$)} = \sqrt{F$-core($I$)}$ for all ideals $I$ in $R$.

To better understand these minimal $cl$-reductions, Epstein mimicked Vrăciu’s definition of $*$-independence in [Vr1] to define $cl$-independence. The elements $x_1, \ldots, x_n$ are said to be $cl$-independent if $x_i \notin (x_1, \ldots, \widehat{x_i}, \ldots, x_n)^{cl}$; for all $1 \leq i \leq n$. Then he further refines the notion to that of strong $cl$-independence. An ideal is strongly $cl$-independent if every minimal set of generators is $cl$-independent. Epstein then showed in [Ep, Proposition 2.3] that when $cl$ is a Nakayama closure, $J$ is a minimal $cl$-reduction of $I$ if and only if $J$ is a strongly $cl$-independent ideal.

In a Noetherian local ring of characteristic $p$ Vrăciu [Vr1] defined the special tight closure, $I^{ssp}$, to be the elements $x \in R$ such that $x \in (mI^{[q_0]})^*$ for some $q_0$. Huneke and Vrăciu show in [Vr1, Proposition 4.2] that $I^{ssp} \cap I = mI$ if $I$ is generated by $*$-independent elements. Note that the minimal $*$-reductions of $I$ are generated by $*$-independent elements. Epstein showed in [Ep, Lemma 3.4] that $I^{ssp} = J^{ssp}$ for all $*$-reductions of $I$.

An ideal $I$ is said to have $cl$-spread, $\ell^{cl}(I)$, if all minimal $cl$ reductions have the same size generating sets. As with the analytic spread, Epstein proves that if $J$ is a minimal $cl$-reduction then $\mu(J) = \ell^{cl}(I)$. He also goes on to prove [Ep, Theorem 5.1] that the $*$-spread is well defined over an excellent analytically irreducible local domain of characteristic $p$. Now if the $cl_1$ and the $cl_2$ spread are defined for $I$, we have:

Proposition 3.7. Let $cl_1$ be a closure operation and $cl_2$ be Nakayama closure operation defined on a Noetherian ring $R$ with $cl_1 \leq cl_2$. Let $I$ be an ideal with well-defined $cl_1$- and $cl_2$-spread then $\ell^{cl_1}(I) \geq \ell^{cl_2}(I)$.
Let $J_1$ be a $c_1$-minimal reduction of $I$. Then $\mu(J_1) = \ell^{c_1}(I)$ ([Ep, Proposition 2.4]). Also $J_1 \subset I \subset J_1^{c_1} \subset J_1^{c_2}$, since $c_1 \leq c_2$. Therefore $J_1$ is also a $c_2$-reduction of $I$ (not necessarily minimal). Hence $\mu(J_1) \geq \ell^{c_2}(I)$ and equality holds if and only if $J_1$ is a minimal $c_2$ reduction of $I$, according to [Ep, Proposition 2.4]. Hence $\ell^{c_1}(I) = \mu(J_1) \geq \ell^{c_2}(I)$. \hfill $\square$ 

In particular, we have the following corollary regarding the Frobenius or $F$-spread, the $*$-spread and the spread of an ideal:

**Corollary 3.8.** Let $R$ be an excellent analytically irreducible local domain of characteristic $p$ then $\ell(I) \leq \ell^*(I) \leq \ell^F(I)$ for all ideals $I$ in $R$.

The spread is bounded by the dimension of the ring, but in principle, the $cl$-spreads can grow arbitrarily large. The $cl$-spread of an ideal $I$ is however bounded by the minimal number of generators of $I$, $\mu(I)$.

There are two invariants of a ring related to the spread: the analytic deviation and the second analytic deviation. Recall that in a Noetherian ring, the analytic deviation of an ideal $I$ is $ad(I) = \ell(I) - ht I$. Note that $I$ is equimultiple if $ad(I) = 0$. The second analytic deviation of $I$ is $ad_2(I) = \mu(I) - \ell(I)$. We make the following definitions with respect to the $cl$-spread of an ideal $I$.

**Definition 3.9.** The $cl$-deviation of an ideal $I$ in a Noetherian ring is $cld(I) = \ell^{cl}(I) - ht I$. The second $cl$-deviation of $I$ is $cld_2(I) = \mu(I) - \ell^{cl}(I)$.

**Remark 3.10.** The following are straightforward from the definition above.

(a) Note that every $m$-primary ideal $I$ is equimultiple, i.e., $ad(I) = 0$. In general, for $cl \leq c$, $cld(I) \geq 0$.

(b) Note that in a Cohen–Macaulay ring, if $I$ is generated by a system of parameters then $I$ is equimultiple and we have $cld(I) = 0$.

(c) Since $\ell(I) \leq \ell^{cl}(I)$, then $cld_2(I) \leq ad_2(I)$.

Note if $I$ is $cl$-closed, then $\ell^{cl}(I) = \mu(I)$. If $I$ is a basic ideal (i.e. $\ast$-basic) and $cl \leq c$, then $\ell^{cl}(I) = \ell(I)$. We would like to know how the $core(I)$ and the $cl$-$core(I)$ are related when $\ell(I) = \ell^{cl}(I)$.

4. When $\ast$-core and core agree

First we record some straightforward cases when the core and the $\ast$-core agree. Since an ideal generated by a system of parameters is both basic and $\ast$-basic we obtain the following:

**Proposition 4.1.** Let $(R, m)$ be a Noetherian ring of characteristic $p$ and $I$ be an ideal generated by a system of parameters, then $\ast$-$core(I) = core(I)$.

**Proof.** A system of parameters is basic and $\ast$-basic, hence, the only reduction (and $\ast$-reduction) of $I$ is $I$. Hence, $\ast$-$core(I) = core(I) = I$. \hfill $\square$
Note, when \( I \) is generated by a system of parameters, we may have \( I^* \subsetneq \mathcal{T} \), but the core and the \(*\)-core are equal.

In a one-dimensional domain with infinite residue field, the integral closure and tight closure of any ideal agree [Hu2, Example 1.6.2]. Hence, we have the following:

**Proposition 4.2.** Let \((R, m)\) be a one-dimensional domain of characteristic \( p \) with infinite residue field, then \(*\)-core\((I) = \text{core}(I)\) for all \( I \subset R \).

**Proof.** If \( I = 0 \) then the assertion is clear. Suppose then that \( I \neq 0 \) then \( \ell(I) = 1 \). By [Hu2, Example 1.6.2] it is known that for principal ideals \((x) = (x)^*\) and also that \( I^* = (x)^* \), for some \( x \in R \). Then every minimal reduction and hence minimal \(*\)-reduction of \( I \) is principal. Therefore we obtain equality of the core and the \(*\)-core. \( \square \)

We would like to show that in an excellent normal local ring the core and the \(*\)-core agree for ideals of second \(*\)-deviation 1. Note that if \((R, m)\) is a Gorenstein local isolated singularity of characteristic \( p > 0 \) with test ideal equal to the maximal ideal and \( I \) is an ideal generated by part of a system of parameters, then \( *d_2(I) = 1 \) by Theorem 2.2(since the tight closure is the socle in this case).

To show that the core and the \(*\)-core agree for ideals with \( *d_2(I) = 1 \), we will consider general minimal reductions. Recall:

**Definition 4.3.** Let \( R \) be a Noetherian local ring with infinite residue field \( k \). Let \( I = (f_1, \ldots, f_m) \) be an \( R \)-ideal and let \( t \) be a fixed positive integer. We say that \( b_1, \ldots, b_t \) are \( t \) general elements in \( I \) if there exists a dense open subset \( U \) of \( A^{tm}_R \) such that for \( 1 \leq j \leq m \), we have \( b_i = \sum_{j=1}^{m} \lambda_{ij} f_j \), where \( \lambda = [\lambda_{ij}]_{ij} \in A^{tm}_k \), \( \lambda \in U \) vary in \( U \) and \( \overline{\lambda} \) is the image of \( \lambda \) in \( A^{tm}_k \). The ideal \( J \) is called a general minimal reduction of \( I \) if \( J \) is a reduction of \( I \) generated by \( \ell(I) \) general elements.

The next two Theorems show that general minimal \(*\)-reductions exist.

**Theorem 4.4.** Let \( R \) be an excellent normal local ring of characteristic \( p > 0 \) with perfect infinite residue field. Let \( I \) be an ideal with \( *d_2(I) = 1 \). Then any ideal generated by \( \ell^*(I) \) general elements of \( I \) is a minimal \(*\)-reduction of \( I \).

**Proof.** Let \( \ell^*(I) = s \). Note that for some \(*\)-independent elements \( f_1, \ldots, f_s \in I \), \( I^* = J^* \) where \( J = (f_1, \ldots, f_s) \). Thus \( J \) is a minimal \(*\)-reduction of \( I \). By [Ep, Lemma 2.2] we know that this generating set of \( J \) can be extended to a generating set of \( I \). In other words, \( I = (f_1, \ldots, f_s, f_{s+1}) \).

Let \( T = R[X_{ij}] \) where \( 1 \leq i \leq s \) and \( 1 \leq j \leq s + 1 \). Let \( a_i = \sum_{j=1}^{s+1} X_{ij} f_j \) for \( 1 \leq i \leq s \) and consider the \( T \)-ideal \( \mathcal{J} = (a_1, \ldots, a_s) \). Write \( \overline{X} \) for \([X_{ij}]_{ij}\).
Consider the $R$-homomorphism $\pi_\lambda : T \to R$ that sends $X$ to $\lambda$, where $\lambda \in A_R^{s(s+1)}$. Notice that for $\lambda_0 = [\delta_{ij}]$ one has $\pi_{\lambda_0}(\tilde{J}) = J$.

Let $m$ denote the maximal ideal of $R$ and $k = R/m$ be the residue field of $R$. We need to find a dense open set $U \subset A_k^{s(s+1)}$, such that $\pi_\lambda(\tilde{J})$ is also a *-reduction for $\lambda \in U$. Let $\lambda_{ij}$ be the image of $\lambda_{ij}$ in $R/m$. Then the generators of the $R/m$ vector space $\pi_\lambda(\tilde{J})/m\pi_\lambda(\tilde{J})$ are $\bar{u}_i = \sum_{j=1}^{s+1} \lambda_{ij} f_j$.

Define $L = [\lambda_{ij}]_{ij}$ to be the matrix defined by the coefficients of the $a_i$, $i = 1, \ldots, s$. $L$ is a $s \times (s+1)$ matrix with coefficients in $k = R/m$. Suppose $L_s$ is the $s \times s$ submatrix of $L$ obtained by omitting the last column. We define the open set $U \subset A_k^{s(s+1)}$ to be set of $L$'s satisfying $\det(L_s) \neq 0$. Since $\{L | \det(L_s) = 0\}$ is closed, then clearly $U$ is open. To see that $U$ is dense, suppose $U'$ is an open set containing a point $\overline{\mu} \notin U$. We need to see that $U \cap U'$ is not empty. Let $M$ be the $s \times (s+1)$ matrix with coefficients in $k = R/m$ representing $\overline{\mu}$. Suppose $M_s$ is the $s \times s$ submatrix of $M$ obtained by omitting the last column. Since $\overline{\mu} \notin U$ then $\det(M_s) = 0$. Since the set of $\overline{\mu}$ with $\det(M_s) = 0$ forms an ideal in $A_k^{s(s+1)}$, then if $U'$ is open, then $U'$ has to contain elements $\overline{\lambda}$ with $\det(L_s) \neq 0$. Hence, $U \cap U' \neq \emptyset$ and $U$ is dense.

Since for any $\overline{\lambda} \in U$, $\pi_\lambda(\tilde{J})$ is a general reduction with $\det(L_s) \neq 0$, then $V = \pi_\lambda(\tilde{J})/m\pi_\lambda(\tilde{J})$ is a $s$-dimensional $k = R/m$-vector space with basis $\overline{\alpha}_1, \ldots, \overline{\alpha}_s$. Row reducing $\overline{\mu}$, we obtain the following matrix:

$$
\begin{pmatrix}
1 & 0 & 0 & \cdots & 0 & \beta_1 \\
0 & 1 & 0 & \cdots & 0 & \beta_2 \\
0 & 0 & 1 & \cdots & 0 & \beta_3 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & 0 & \cdots & 1 & \beta_s
\end{pmatrix}
$$

where the $\beta_i \in k$. This implies that an alternate basis for $V$ is $\{f_1 + \beta_1 f_{s+1}, \ldots, f_s + \beta_s f_{s+1}\}$. Let $J_{\text{gen}} = (f_1 + \beta_1 f_{s+1}, \ldots, f_s + \beta_s f_{s+1}) = \pi_\lambda(\tilde{J})$.

Case 1: Suppose that for all $1 \leq i \leq s$ we have that $\beta_i \in m$. Let $K = J_{\text{gen}} + mI$. Then we claim that $K = J + mI$. To see this consider a generator $\alpha$ for $K$. Then $\alpha = f_1 + \beta_1 f_{s+1} + \delta$, where $\delta \in mI$. But as $\beta_i \in m$ and $f_i \in I$ then $\alpha \in J + mI$. Now let $\alpha'$ be a generator of $J + mI$. Then $\alpha' = f_i + \delta'$, where $\delta' \in mI$. Since $\beta_i \in m$ then $\delta' = \beta_i f_{s+1} \in mI$. Hence $\alpha' = f_i + \beta_i f_{s+1} + (\delta' - \beta_i f_{s+1}) \in J_{\text{gen}} + mI = K$.

Next we claim that $(J + mI)^* = J^*$. Notice that $J \subset J + mI \subset I$. Taking the tight closure we obtain $J^* \subset (J + mI)^* \subset I^* = J^*$. Thus, $(J + mI)^* = J^*$. Overall we have the following inclusions:

$$
J_{\text{gen}} \subset I \subset (J + mI)^* = (J_{\text{gen}} + mI)^*
$$
Now by [Ep, Proposition 2.1] we have that $I^* \subset J_{gen}^*$.

Case 2: If $\beta_i \not\in m$ for some $i$ then without loss of generality we may assume that $i = s$ and $\beta_s = 1$. Then $J_{gen} = (f_1 + \beta_1 f_{s+1}, \ldots, f_s + f_{s+1})$. Hence $f_1 - \beta_1 f_s \in J_{gen}$. Let $f'_1 = f_1 - \beta_1 f_s$ and replace $f_1$ with $f'_1$. Continuing this way we may assume that $J_{gen} = (f_1, \ldots, f_{s-1}, f_s + f_{s+1})$. Suppose that $f_{s+1} \not\in (f_1, \ldots, f_{s-1}, f_s + f_{s+1})^*$. Let $f'_{s+1} = 1 - \beta_1 f_s + f_{s+1}$. Thus $(f_1, \ldots, f_{s-1}, f_s, f_{s+1}) \subset J_{gen}$. Therefore $f_1 - \beta_1 f_s \not\in J_{gen}$.

Since $f_{s+1} \in I \subset J^*$, then we may take $c \in R^0$ such that $cf_{s+1}^q \in J^q$ for every $q = p^\ell$. Hence $cf_{s+1}^q = \sum_{i=1}^s r_i f_i^q$, then $cf_{s+1}^q + r_{sq} f_{s+1}^q = \sum_{i=1}^{s-1} r_i f_i^q + r_{sq}(f_{s} + f_{s+1}^q)$. Let $c = c + r_{sq}$. Then $c f_{s+1}^q = \sum_{i=1}^{s-1} r_i f_i^q + r_{sq}(f_{s} + f_{s+1}^q)$ and in particular $c f_{s+1}^q \in (f_1, \ldots, f_{s-1}, f_s + f_{s+1})^q$. Therefore by [Ab, Proposition 2.4] $c q \in m^{[N(q)]}$. For every integer $N$ there exists a $q_N$ such that $q_N \in m^N$. Fix $N_0$ such that $c q \not\in m^{N_0}$. It then follows that for every $N \geq N_0$ we have $c + r_{sq} \in m^N$. Since $c q \not\in m^N$ then $r_{sq} \not\in m^N$ for all $N \geq N_0$.

Notice that $c f_{s+1}^q - \sum_{i=1}^{s-1} r_{iq} f_i^q = r_{sq} f_{s+1}^q$. Thus by [Ab, Proposition 2.4] we have that $f_s \in (f_1, \ldots, f_{s-1}, f_{s+1})^*$ since $r_{sq} \not\in m^N$. Therefore $(f_1, \ldots, f_{s-1}, f_{s+1})^* = I^*$ and thus $(f_1, \ldots, f_{s-1}, f_{s+1})$ is a minimal $*$-reduction of $I$.

By [HV, Theorem 2.1] we have that $J^* = J + J^{*p}$. Also by [Ep, Lemma 3.4] since $J \subset I$ and $J^* = I^*$ then $J^{*p} = I^{*p}$. Therefore $I^* = J + I^{*p}$. Since $f_{s+1} \in I^*$ and $f_{s+1} \not\in J$ then $f_{s+1} \not\in I^{*p}$. Therefore $(f_1, \ldots, f_{s-1}, f_{s+1})$ is not a minimal $*$-reduction of $I$, by [Vr2, Proposition 1.12(b)], which is a contradiction. Therefore $f_{s+1} \in J_{gen}^*$ and thus $J_{gen}^* = I^*$.

We are able to generalize Theorem 4.4 and relax the condition on $d_2(I)$.

**Theorem 4.5.** Let $R$ be an excellent normal local ring of characteristic $p > 0$ with perfect infinite residue field. Let $I$ be an ideal. Then any ideal generated by $\ell^*(I)$ general elements of $I$ is a minimal $*$-reduction of $I$.

**Proof.** Let $\ell^*(I) = s$. Then there exists $*$-independent elements $f_1, \ldots, f_s \in I$ such that $I^* = (f_1, \ldots, f_s)^*$. Let $J = (f_1, \ldots, f_s)$. Hence $J$ is a minimal $*$-reduction of $I$. By [Ep, Lemma 2.2] we know that this generating set of $J$ can be extended to a generating set of $I$. In other words, $I = (f_1, \ldots, f_s, f_{s+1}, f_{s+2}, \ldots, f_{s+n})$.

As above we form an ideal generated by general elements and we may assume that $J_{gen} = (f_1 + \beta_1 f_{s+1}, \ldots + \beta_1 f_{s+n}, \ldots, f_s + \beta_s f_{s+1} + \ldots + \beta_s f_{s+n})$. Let $m$ be the maximal ideal of $R$ and $k = R/m$ be the residue field of $R$.

Case 1: Suppose that for all $1 \leq i \leq s$ and for all $1 \leq j \leq n$ we have that $\beta_{ij} \in m$. Let $K = J_{gen} + mI$. Then we claim that $K = J + mI$. To see this consider a generator $\alpha$ for $K$. Then $\alpha = f_i + \beta_i f_{s+1} + \ldots + \beta_i f_{s+n} + \delta$, where $\delta \in mI$. But as $\beta_{ij} \in m$ and $f_i \in J$ then $\alpha \in J + mI$. Now let $\alpha'$ be a generator of $J + mI$. Then $\alpha' = f_i + \delta'$, where $\delta' \in mI$. Since $\beta_{ij} \in m$ for all $1 \leq i \leq s$ and for all $1 \leq j \leq n$ then $\delta' - \beta_{ij} f_{s+1} + \ldots + \beta_{ij} f_{s+n} \in mI$. Hence $\alpha' = f_i + \beta_{ij} f_{s+1} + \ldots + \beta_{ij} f_{s+n} + (\delta' - \beta_{ij} f_{s+1} + \ldots + \beta_{ij} f_{s+n}) \in J_{gen} + mI = K$. 

Next we claim that \((J + \mathfrak{m}I)^* = J^*\). Notice that \(J \subset J + \mathfrak{m}I \subset I\). Taking the tight closure we obtain \(J^* \subset (J + \mathfrak{m}I)^* \subset I^* = J^*\). Thus, \((J + \mathfrak{m}I)^* = J^*\). Overall we have the following inclusions:

\[
J_{\text{gen}} \subset I \subset (J + \mathfrak{m}I)^* = (J_{\text{gen}} + \mathfrak{m}I)^*
\]

Now by [Ep, Proposition 2.1] we have that \(I^* \subset J_{\text{gen}}^*\).

Case 2: Suppose \(\beta_{ij} \not\in \mathfrak{m}\) for some \(1 \leq i \leq s\) and \(1 \leq j \leq n\). Without loss of generality we may assume that \(i = s\) and \(j = n\) and that \(\beta_{sn} = 1\). Hence \(J_{\text{gen}} = (f_1 + \beta_{11}f_{s+1} + \ldots + \beta_{1n}f_{s+n}, \ldots, f_s + \beta_{sn}f_{s+1} + \ldots + f_{s+n})\).

We will proceed by induction on \(n = *d_2(I) = \mu(I) - \ell^*(I)\). If \(n = 0\) there is nothing to show and if \(n = 1\) then Theorem 4.4 gives the result. So we assume that \(n > 1\) and for the results holds for any ideal \(I'\) with \(*d_2(I') = n - 1\).

Let \(g_i = f_i + \beta_{i1}f_{s+1} + \ldots + \beta_{in}f_{s+n}\). Then \(g_i - \beta_{in}g_s = (f_i - \beta_{in}f_s) + \sum_{j=1}^{n-1}(\beta_{ij} - \beta_{in}\beta_{sj})f_{s+j} \in J_{\text{gen}}\). Notice that \(f_i' = f_i - \beta_{in}f_s \in J\) and let \(\beta_{ij}' = \beta_{ij} - \beta_{in}\beta_{sj}\). Therefore, we can replace \(f_i\) with \(f_i'\) and replace \(\beta_{ij}\) with \(\beta_{ij}'\) to assume that \(J_{\text{gen}} = (f_1 + \beta_{11}f_{s+1} + \ldots + \beta_{1(n-1)}f_{s+n-1}, \ldots, f_s + \beta_{s1}f_{s+1} + \ldots + \beta_{s(n-1)}f_{s+n-1}, f_s + \beta_{sn}f_{s+1} + \ldots + f_{s+n})\).

Let \(h_i = f_i + \beta_{i1}f_{s+1} + \ldots + \beta_{i(n-1)}f_{s+n-1}\). Then \(J_{\text{gen}} = (h_1, \ldots, h_s, f_s + f_{s+n})\). Let \(L = (h_1, \ldots, h_s)\) and \(J_1 = (f_1, \ldots, f_s, f_{s+1}, \ldots, f_{s+n-1})\).

Since \(g_i\) is a general element for all \(1 \leq i \leq s\) then there exists \(U \subset \mathbb{A}^s_{k(s+n)}\) a dense open such that the image of \(\beta_i = [0, \ldots, 0, 1, 0, \ldots, \beta_{i1}, \ldots, \beta_{i(n-1)}, \beta_{im}]\) varies in \(U\). Consider the natural projection map \(\pi : \mathbb{A}^s_{k(s+n)} \to \mathbb{A}^s_{k(s+n-1)}\) such that \(\pi((a_1, \ldots, a_{s+n-1}, a_{s+n})) = (a_1, \ldots, a_{s+n-1})\) and \(a_i \in \mathbb{A}^s_{k}\). Let \(W = \pi(U)\). As \(U\) is dense and open then \(U \not= \emptyset\) and thus \(W \not= \emptyset\) and \(W\) is also open, since \(\pi\) is an open map. Therefore \(W\) is a dense open set. As \(\beta_i\) is allowed to vary in \(U\) then \(\pi(\beta_i)\) varies in \(W\) and thus \(h_i\) is also a general element.

Notice that \(J \subset J_1 \subset I^*\) and thus \(J^* = J^*_1\). Hence \(\ell^*(J_1) \leq s\). Therefore \(*d_2(J_1) \leq n - 1\) and hence by our inductive hypothesis \(L\) is a \(*\)-reduction of \(J_1\) and \(L^* = J^*_1 = J^*\). We are claiming that \(J^*_{\text{gen}} = L^* = I^*\). It is enough to show that \(f_{s+n} \in J^*_{\text{gen}}\). Suppose that \(f_{s+n} \not\in J^*_{\text{gen}}\). Then as in the proof of Theorem 4.4 we obtain that \(h_s \in (h_1, \ldots, h_{s-1}, f_{s+n})^*\). By [HV, Theorem 2.1] we have that \(L^* = L + L^{*p}\). Also by [Ep, Lemma 3.4] since \(L \subset I\) and \(L^* = I^*\) then \(L^{*p} = I^{*p} = J^{*p}\). Therefore \(I^* = L + I^{*p}\). Since \(f_{s+n} \in I^*\) and \(f_{s+n} \not\in L\) then \(f_{s+n} \not\in I^{*p}\). Therefore \((h_1, \ldots, h_{s-1}, f_{s+n})\) is not a minimal \(*\)-reduction of \(I\), by [Vr2, Proposition 1.12(b)], which is a contradiction. Hence \(f_{s+n} \in J^*_{\text{gen}}\) and \(J^*_{\text{gen}} = L^* = I^*\).

\(\square\)

**Corollary 4.6.** Let \((R, \mathfrak{m})\) be a Gorenstein isolated singularity of characteristic \(p > 0\) with perfect infinite residue field. Suppose that the test ideal of \(R\) is \(\mathfrak{m}\). Let \(I\) be a tightly closed ideal and suppose that \(I = J^*\) where \(J\) is a parameter ideal minimally generated by \(s\) elements. Then any ideal generated by \(s\) general elements of \(I\) is a minimal \(*\)-reduction of \(I\).
Proof. Suppose $J = (f_1, \ldots, f_s)$ is a parameter ideal. Then $J$ is a minimal $*$-reduction of $I = J^* = (J : m)$, where the last equality is obtained by Theorem 2.2. Since $R$ is Gorenstein then the socle $(J : m)/J$ is a one dimensional vector space. Hence $I = (f_1, \ldots, f_s, f_{s+1})$, where $f_{s+1} \notin J$. Therefore $\mu(I) = s + 1$ and $*d_2(I) = 1$. Thus by Theorem 4.4, any ideal generated by $s$ general elements is a minimal $*$-reduction of $I$. □

Corollary 4.7. Let $(R, m)$ be a $d$-dimensional Gorenstein isolated singularity of characteristic $p > 0$ with perfect infinite residue field. Suppose that the test ideal of $R$ is $m$. Let $I$ be an $m$-primary tightly closed ideal and suppose that $I = J^*$ where $J$ is a parameter ideal. Then $\text{core}(I) = \text{core}^*(I)$.

Proof. Since $I$ is $m$-primary then $\ell(I) = \ell^*(I) = d$. By Corollary 4.6 any ideal generated by $d$ general elements is a general minimal $*$-reduction. Notice that these minimal general $*$-reductions are also minimal reductions of $I$, since $\ell(I) = d$.

Also, since $I$ is $m$-primary then by Theorem 2.6 ([CPU1, Theorem 4.5]) we have that $\text{core}(I)$ is a finite intersection of general minimal reductions. Since each general minimal reduction is also a minimal $*$-reduction then $\text{core}^*(I) \subset \text{core}(I)$. On the other hand $\text{core}(I) \subset \text{core}^*(I)$, by Corollary 3.5. □

Theorem 4.8. Let $R$ be a local Cohen–Macaulay normal domain of characteristic $p > 0$ with perfect infinite residue field. Let $I$ be an ideal with $\ell^*(I) = \ell(I) = s$. We further assume that $I$ satisfies $G_s$ and is weakly $(s - 1)$-residually $S_2$. Then $\text{core}(I) = \text{core}^*(I)$.

Proof. We know that $\text{core}(I) \subset \text{core}^*(I)$ by Corollary 3.5. According to Theorem 2.6 the core is a finite intersection of general minimal reductions. Since every general minimal reduction is a minimal $*$-reduction by Theorem 4.5, we obtain the opposite inclusion. □

5. The $*$-core in complete one dimensional semigroup rings

In Proposition 4.1, we saw that the core and the $*$-core agree for all ideals in a one dimensional domain of characteristic $p > 0$ with infinite residue field. In Huneke and Swanson’s paper [HS2], one of the first questions that they ask is: if $I$ is integrally closed, is $\text{core}(I)$ integrally closed? They settle this question in the setting of a two-dimensional regular local ring. Corso, Polini and Ulrich in [CPU2, Theorem 2.11] showed that if $R$ is a local Cohen–Macaulay normal ring with infinite residue field then $\text{core}(I)$ is integrally closed, when $I$ is a normal ideal of positive height, universally weakly $(\ell - 1)$-residually $S_2$ and satisfies $G_\ell$ and $AN_{\ell - 1}$, where $\ell = \ell(I)$. A related question is: if $I$ is tightly closed, is $\text{core}^*(I)$ tightly closed? We will consider this question now for complete one-dimensional semigroup rings with test ideal equal to the maximal ideal. The second author showed the following:
Theorem 5.1. ([Va]) Let \((R, m)\) be a one-dimensional domain. The test ideal of \(R\) is equal to the conductor, i.e. \(\tau = c = \{c \in R | \phi(1) = c, \phi \in \text{Hom}_R(\Omega, R)\}\).

Note, in a one-dimensional local semigroup ring, the semigroup is a sub-semigroup of \(N_0\). For each sub-semigroup \(S \subseteq N_0\), there is a smallest \(m\) such that for all \(i \geq m, i \in S\). The conductor of such a one dimensional semigroup ring is \(c = \langle t^m, t^{m+1}, t^{m+2}, \ldots \rangle\), [Ei, Exercise 21.11]. Hence, the test ideal in a one-dimensional semigroup ring is the maximal ideal, if the conductor is the maximal ideal. This can only happen if the semigroup has the form \(c = \langle n \cdot 1 \rangle\).

Proposition 5.2. Each nonzero nonunit principal ideal of \(R = k[[t^n, t^{n+1}, \ldots, t^{2n-1}]]\) can be expressed in the form \((t^m + a_1 t^{m+1} + \cdots + a_{n-1} t^{m+n-1})\), \(a_i \in k, m \geq n\).

**Proof.** Suppose \(0 \neq f \in R\). Thus, after multiplying by a nonzero element of \(k\), \(f = t^m + a_1 t^{m+1} + a_2 t^{m+2} + \cdots \) for \(m \geq n\). We will show that \(t^r \in (f)\) for \(r \geq m + n\). Hence, \(t^m + a_1 t^{m+1} + \cdots + a_{n-1} t^{m+n-1} \in (f)\). Similarly, \(t^r \in (t^m + a_1 t^{m+1} + \cdots + a_{n-1} t^{m+n-1})\) for \(r \geq m + n\). Hence, \(f \in (t^m + a_1 t^{m+1} + \cdots + a_{n-1} t^{m+n-1})\).

Let \(g \in k[[t]]\). Note that \(t^{r-m} g \in k[[t^n, t^{n+1}, t^{2n-1}]]\). Hence, if \(g\) is a unit in \(k[[t]]\), then \(t^{r-m} g^{-1} \in k[[t^n, t^{n+1}, t^{2n-1}]]\) also. In \(k[[t]]\),

\[ f = t^m(1 + a_1 t + a_2 t^2 + \cdots) = t^m g. \]

Note that \(t^{r-m} g^{-1} f = t^r\). Similarly \(t^r \in (t^m + a_1 t^{m+1} + \cdots + a_{n-1} t^{m+n-1})\). Since \(f - (t^m + a_1 t^{m+1} + \cdots + a_{n-1} t^{m+n-1}) = a_n t^{2n} + a_{n+1} t^{2n+1} + \cdots\) which is an element of \((f) \cap (t^m + a_1 t^{m+1} + \cdots + a_{n-1} t^{m+n-1})\), we see that \((t^m + a_1 t^{m+1} + \cdots + a_{n-1} t^{m+n-1}) = (f)\). Hence, all principal ideals of \(k[[t^n, t^{n+1}, t^{2n-1}]]\) have the form \((t^m + a_1 t^{m+1} + \cdots + a_{n-1} t^{m+n-1})\).

\[ \square \]

Proposition 5.3. Let \(k\) be an infinite field of characteristic \(p > 0\). Any tightly closed ideal in \(R = k[[t^n, t^{n+1}, t^{2n-1}]]\) is of the form \((t^m, t^{m+1}, \ldots, t^{m+n-1})\) for some \(m \geq n\).

**Proof.** Suppose \(I\) is a tightly closed ideal in \(R\). Since \(R\) is a one dimensional domain, there is a principal ideal \((f) \in I\), with \((f)^* = I\). By Proposition 5.2,

\[ (f) = (t^m + a_1 t^{m+1} + \cdots + a_{n-1} t^{m+n-1}) \]

for some \(m \geq n\) and \(a_i \in k\). Using Theorem 2.2 and the arguments we followed in Proposition 5.2, \(I = (f)^* = (f) : m = (t^m, t^{m+1}, \ldots, t^{m+n-1})\). \(\square\)

Proposition 5.4. Let \(R = k[[t^n, t^{n+1}, t^{2n-1}]]\) with \(k\) an infinite field of characteristic \(p > 0\). If \(I \subset R\) is tightly closed, then \(*\text{-core}(I)\) is tightly closed.
Proof. If $I = (0)$, then clearly $\ast\text{-core}(I) = (0)$ and thus the assertion is clear. Since $R$ is a one-dimensional domain then $\text{core}(I) = \ast\text{-core}(I)$. If $I$ is basic then $I$ is also $\ast$-basic and again the assertion is clear. So suppose $I$ is not basic, nonzero and tightly closed. Then $I = (t^m, t^{m+1}, \ldots, t^{m+n-1})$ for some $m \geq n$, by Proposition 5.3. Since $I$ is non-zero then $I$ is $\mathfrak{m}$-primary, where $\mathfrak{m}$ is the maximal ideal of $R$. Hence by Theorem 2.6 we have that $\text{core}(I) = \bigcap_{i=1}^{s} (f_i)$, for some positive integer $s$ and $(f_i)$ general minimal reductions of $I$ for all

$1 \leq i \leq s$. Let $(f_i)$ be such a general minimal reduction. Then $(f_i) = (t^m + a_{i1}t^{m+1} + \cdots + a_{i(n-1)}t^{m+n-1})$ for some $a_{ij} \in k$, since $f_i$ is a general element in $I$. As in the proof of Proposition 5.2, we see that $t^r \in (t^m + a_{i1}t^{m+1} + \cdots + a_{i(n-1)}t^{m+n-1})$ for all $r \geq m + n$. Hence, $(t^{m+n}, t^{m+n+1}, \ldots, t^{m+2n-1}) \subset (f_i)$ for all $i$ and thus $(t^{m+n}, t^{m+n+1}, \ldots, t^{m+2n-1}) \subset \ast\text{-core}(I)$.

On the other hand let $g \in \ast\text{-core}(I)$. Hence $g \in \bigcap_{i=1}^{s} (f_i)$. It is clear that $(g) \neq (f_i)$ for some $i$. Then $g = a(t^m + a_{i1}t^{m+1} + \cdots + a_{i(n-1)}t^{m+n-1})$ for some $a \in R$ and $a_{ij} \in k$. If $a$ is a unit then $(g) = (f_i)$, which is a contradiction. Hence we may assume that $a$ is not a unit. Thus $a = \beta_1t^n + \beta_2t^{n+1} + \cdots$ and $g = \gamma_0t^{m+n} + \gamma_1t^{m+n+1} + \cdots \gamma_{n-1}t^{m+2n-1} + t^{m} \gamma_{n} t^{m+n} + \gamma_{n+1}t^{m+n+1} + \cdots \gamma_{2n-1}t^{m+2n-1} + \cdots$. Therefore $g \in (t^{m+n}, t^{m+n+1}, \ldots, t^{m+2n-1})$ and thus $\ast\text{-core}(I) \subset (t^{m+n}, t^{m+n+1}, \ldots, t^{m+2n-1})$. Finally notice that $(t^{m+n}, t^{m+n+1}, \ldots, t^{m+2n-1})$ is a tightly closed ideal and $\ast\text{-core}(I) = (t^{m+n}, t^{m+n+1}, \ldots, t^{m+2n-1})$.

Note, since the $\text{core}(I) = \ast\text{-core}(I)$ by Proposition 4.1 and the tight closure of an ideal agrees with the integral closure in a one dimensional domain with infinite residue field, we obtain:

**Corollary 5.5.** Let $R = k[[t^n, t^{n+1}, \ldots, t^{2n-1}]]$ with $k$ an infinite field of characteristic $p > 0$. If $I \subset R$ is integrally closed, then $\text{core}(I)$ is integrally closed.

**Remark 5.6.** The question of whether the core of an integrally closed ideal is integrally closed as well was first addressed by Huneke and Swanson, [HS1]. They answer this question positively when the ring is a 2-dimensional regular ring, [HS1, Corollary 3.12]. This question was also addressed by several other authors later, (see [CPU2, Theorem 2.11, Corollary 3.7], [PU, Corollary 4.6], and [HyS1, Proposition 5.5.3]).

We note here that Corollary 5.5 is not covered by any of the results mentioned above. In [CPU2, Corollary 3.7] and [PU, Corollary 4.6] it is required that the ring $R$ is Gorenstein. The ring $R = k[[t^n, t^{n+1}, \ldots, t^{2n-1}]]$ with $k$ an infinite field of characteristic $p > 0$ is not Gorenstein unless $n = 2$. In [CPU1, Theorem 2.11] the Gorenstein condition can be relaxed to Cohen–Macaulay rings, but in addition the Rees algebra of $I$ and $I$ are assumed to be normal and $J : I$ is independent of $J$ for
every minimal reduction $J$ of $I$. Notice that the ideal $I$ in Corollary 5.5 is normal and $J : I = \tau = m$ is independent of the minimal reduction $J$. However, the Rees algebra of $I$ is not normal, since $R$ is not normal. Finally, in [HyS1, Proposition 5.5.3] it is assumed that the ring $R$ is Cohen–Macaulay, $R$ contains the rational numbers and the Rees algebra of $I$ is Cohen–Macaulay whereas the ring in Corollary 5.5 does not contain the rational numbers.

6. Examples

Since, the tight closure of an ideal is much closer to the ideal than the integral closure, we expected to find examples of ideals $I$ where the $\ast$-core$(I) \supseteq \text{core}(I)$. The following example gives a family of rings where $\ast$-core$(m^2) \neq \text{core}(m^2)$.

Example 6.1. Let $R = \mathbb{Z}/p\mathbb{Z}(u, v, w)[[x, y, z]]/(ux^p + vy^p + wz^p)$. Then $R$ is a normal domain, [Ep]. In [Vl], Vraciu and the second author computed the test ideal of $R$ to be $m^{p-1}$, where $m$ is the maximal ideal of $R$.

For $p = 2$, we compute the $\ast$-core of $m^2$. In this case the test ideal is $m$. Note that the $\ast$-spread of $m^2$ is 3. For example $J = (y^2, yz, z^2)$ is a minimal $\ast$-reduction of $m^2$. To see this notice that $x^2 = \frac{v}{u}y^2 + \frac{w}{u}z^2 \in J$ and $(xz)^2 = (\frac{v}{u}y^2 + \frac{w}{u}z^2)^2 = \frac{u}{v}(yz)^2 + \frac{w}{u}(z^2)^2$. Hence $xz \in J^F \subset J^*$. Similarly $xy \in J^*$ and thus $m^2 = J^*$. On the other hand $I = (y^2, z^2)$ is not a $\ast$-reduction of $m^2$ since $J^* = I : m = (y^2, z^2, xyz) \neq m^2$. Similarly $(y^2, yz)$ and $(yz, z^2)$ are not $\ast$-reductions of $m^2$ and thus $\ell^\ast(m^2) = 3$.

In addition we note that $(x^2, xy, y^2), (x^2, xz, z^2), (y^2, yz, z^2), (yz, xz, xy)$ are all minimal $\ast$-reductions of $m^2$. Hence $\ast$-core$(m^2) \subset (x^2, xy, y^2) \cap (x^2, xz, z^2) \cap (y^2, yz, z^2) \cap (yz, xz, xy) = (x^2, y^2, z^2, xyz) \cap (yz, xz, xy) = m^3$. Note that $m^3 = m.J^* \subset J$ for all $J$ minimal $\ast$-reductions of $m^2$, since $m$ is the test ideal. Hence, $\ast$-core$(m^2) = m^3$.

For $p \geq 3$, the computation of the $\ast$-core of $m^2$ is as follows: The $\ast$-spread of $m^2$ is 3. Once again, $J = (y^2, yz, z^2)$ is a minimal $\ast$-reduction of $m^2$. Notice that $(x^2)^p = (x^2)^2 = (\frac{v}{u}y^2 + \frac{w}{u}z^2)^2 = \frac{v^2}{u^2}(y^2)^2 + 2\frac{vw}{u^2}(yz)^2 + \frac{w^2}{u^2}(z^2)^2 \in J^F$ and $(xz)^p = (\frac{v}{u}y^p + \frac{w}{u}z^p)^2$. Thus $xz \in J^F \subset J^*$. Similarly $xy \in J^*$ and therefore $J^* = m^2$. Note that $I = (y^2, z^2)$ is not a $\ast$-reduction of $m^2$ since $I^* = I : m^{p-1} = (y^2, z^2, x^{p-1}yz) : m^{p-2} = (y^2, z^2, x^{p-1}y, x^{p-3}z, x^{p-2}yz) : m^{p-3} = \cdots = (y^2, z^2) + m^3 \neq m^2$. As the test ideal is $m^{p-1}$, we see that $m^{p-1}.J^* = m^{p+1} \subset J$ for all minimal $\ast$-reductions $J$ of $m^2$. Notice also that $(x^2, xy, y^2), (x^2, xz, z^2), (y^2, yz, z^2), (yz, xz, xy)$ are all minimal $\ast$-reductions of $m^2$. Therefore $\ast$-core$(m^2) \subset (x^2, xy, y^2) \cap (x^2, xz, z^2) \cap (y^2, yz, z^2) \cap (yz, xz, xy) = m^{p+1} + (xyz, x^2 y^2, x^2 z^2, y^2 z^2)$. Hence $m^{p+1} \subset \ast$-core$(m^2) \subset m^{p+1} + (xyz, x^2 y^2, x^2 z^2, y^2 z^2)$.

We also note that the $\ell(m^2) = 2$, and that $H = (x^2, yz)$ is a minimal reduction of $m^2$ in any characteristic.

If $p = 2$ then the reduction number of $m^2$ with respect to $H$ is 1. Since $\text{char} k = 2 > 1$ then we may use the formula for the core as in [PU, Theorem 4.5]. Hence
core(m^2) = H^2 : m^2 = m^4, where the last equality follows from calculations using the computer algebra program Macaulay 2, [M2]. Therefore core(m^2) \subseteq \ast\text{-core}(m^2).

If p = 3 then the reduction number of m^2 with respect to \(H\) is 2. Since now char \(k = 3 > 2\) we may again use the formula as in [PU, Theorem 4.5]. Thus core(m^2) = H^3 : m^4 = m^5, where the last equality is again obtained using the computer algebra program Macaulay 2, [M2]. Notice that since \(m^4 \subset \ast\text{-core}(m^2)\) then core(m^2) \subseteq \ast\text{-core}(m^2) again.

When the spread and the \(\ast\)-spread agree, it is not necessarily the case that all reductions of an ideal are \(\ast\)-reductions. However, the following example exhibits that even so, the core and the \(\ast\)-core agree for the maximal ideal in the following ring. In some sense, the following example prompted us to prove Theorem 4.4, Theorem 4.5 and Theorem 4.8.

**Example 6.2.** Let \(R = k[[x, y, z]]/(x^2 - y^3 - z^7)\), where the \(k\) is an infinite field and char \(k > 7\). Let \(m = (x, y, z)\) denote the maximal ideal of \(R\). We observe first that \(m\) is the test ideal, [Va1].

We will show that \(\ast\)-spread of \(m\) is 2, \(\ell(m) = 2\) and core(m) = \(m^2 = \ast\text{-core}(m)\).

First note that \(R\) is a 2–dimensional Gorenstein local ring and hence \(\ell(m) = 2\). Let \(J = (y, z)\). Then \(J\) is a reduction of \(m\) with reduction number 1. Since char \(k > 1\) then core(m) = \(J^2 : m = m^2\) by [PU, Theorem 4.5]. Notice that this does not agree with the formula in Hyry-Smith [HyS2, Theorem 4.1] or Fouli-Polini-Ulrich [FPU, Theorem 4.4] since \(a = 42 - 21 - 14 - 6 = 1\) and core(m) \(\neq m^{2a+1} = m^4\). The hypothesis that \(m\) is generated by elements of degree 1 is important in their formula.

On the other hand, \(J\) is also a minimal \(\ast\)-reduction of \(m\). Note that \(y, z\) form a system of parameter, hence by Theorem 2.2 \((y, z)^* = (y, z) : m = (x, y, z) = m\). Therefore \(\ell^*(m) = 2 = \ell(m)\). We claim that \(J_1 = (x + z, y)\) and \(J_2 = (x + y, z)\) are also minimal \(\ast\)-reductions. Denote \(p_n(x, y) = x^n + x^{n-1}y + \ldots + xy^{n-1} + y^n\). Note that if \(n\) is odd, \(x^n + y^n = (x + y)p_{n-1}(x, -y)\).

Now we can see that \((x + z)p_6(x, -z) + y^3 = x^7 + z^7 + x^2 - y^3 + y^3 = x^2(1 + x^5)\). Since \((1 + x^5)\) is a unit in \(R\), then \(x^2 \in (x + z, y)\). Since \(x(x + z) = x^2 + xz\) we also observe that \(xz \in (x + z, y)\) and similarly, we see that \(z^2 \in (x + z, y)\). Hence \(m^2 \subset (x + z, y)\) and thus \(m^* \subset (x + z, y) : m = (x + z, y)^* \subset m\), i.e. \((x + z, y)^* = m\). Using the same argument exchanging \(y\) and \(z\) and exchanging the powers 3 and 7, we see that \(J_2\) is a minimal \(\ast\)-reduction of \(m\).

Let \(K\) be a minimal \(\ast\)-reduction of \(m\). Then \(m = K^*\). For all minimal reductions \(K, m^2 = mK^* \subset K\). Thus \(m^2 \subset \ast\text{-core}(m)\). We can easily see that \(m^2 = J \cap J_1 \cap J_2\). We can then conclude that \(m^2\) in fact is \(\ast\text{-core}(m)\) and core(m) = \(\ast\text{-core}(m)\).
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References


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