Banach Spaces II

1. (Dual Space) Suppose $V$ is a Banach space and let

$$V^* = \{\text{bounded linear functionals } \lambda : V \to \mathbb{R} \}.$$ 

If $\lambda : V \to \mathbb{R}$ is a bounded linear functional, let

$$|\lambda| = \sup \left\{ \frac{|\phi(v)|}{|v|} : v \neq 0 \right\}.$$ 

Show that $V^*$ with this norm is a Banach space, called the dual space to $V$.

2. (Examples of Duals) Let $l^\infty(\mathbb{R})$ be the space of bounded infinite sequences of real numbers, $c_0(\mathbb{R})$ those sequences in $l^\infty(\mathbb{R})$ which converge to zero, and $l^1(\mathbb{R})$ those sequences which are absolutely convergent, that is $\sum |x_i| < \infty$. Show the following:

a. $l^1(\mathbb{R})$ can be viewed as the dual of $c_0(\mathbb{R})$.

b. $l^\infty(\mathbb{R})$ can be viewed as the dual of $l^1(\mathbb{R})$.

3. Here we will sketch the argument for identifying another very important dual space, namely the dual of the space of continuous functions $C([0,1])$. A function $g : [0,1] \to \mathbb{R}$ is said to be of bounded variation if there is a real number $M$ so that for all finite partitions

$$0 = x_0 < x_1 < \ldots < x_n = 1$$

we have

$$\sum_{i=1}^{n} |g(x_i) - g(x_{i-1})| < M.$$ 

a. If $g$ is of bounded variation and $x \in [0,1]$ show that both right and left limits of $f$ at $x$ exist. Thus $g$ has only very simple “jump” discontinuities.

b. Show that the discontinuities of $g$ are countable in number by parametrizing them according to the size of the jump.

c. The Riemann–Stieltjes integral relative to $g$ is defined by

$$\int_{0}^{1} f dg = \lim_{n} \sum_{i=1}^{n} (f(t_i))(g(x_i) - g(x_{i-1}))$$

where $0 = x_0 < x_1 < \ldots < x_n = 1$ and $t_i \in [x_{i-1},x_i]$; the limit is over all partitions of $[0,1]$ as the size (that is the maximum subinterval length) of the partition goes to zero. Show that if $g(x) = x$ then the Riemann–Stieltjes integral is just the regular Riemann integral.
d. If $$g(x) = \begin{cases} 0 & \text{if } x < x_0 \\ 1 & \text{if } x \geq x_0 \end{cases}$$ what is the associated Riemann–Stieltjes integral?

e. What about $$h(x) = \begin{cases} 0 & \text{if } x \leq x_0 \\ 1 & \text{if } x > x_0 \end{cases}$$

d. These last two problems show that there is not a one-to-one correspondence between elements of $$C([0,1])$$ and elements of $$BV([0,1])$$. One way to remedy this situation is to define a new space $$BV'([0,1])$$ consisting of functions of bounded variation which take the value 0 at 0 and are everywhere left continuous. With this new space, the correspondence is one-to-one as desired but there is a lot work involved. Think about why this is a difficult problem.

4. A Banach space is called reflexive if, like a Hilbert space, it is isomorphic to its double dual. Show that neither $$C([0,1])$$ (with the sup norm) nor $$L^1([0,1])$$ is reflexive.

5. (Open mapping theorem) A continuous surjective linear map $$\phi : V \to W$$ of Banach spaces is open. One tool in the proof of this important result is a purely set theoretic result, the Baire theorem which says that if $$X$$ is a complete metric space and $$\{S_n\}_{n \geq 1}$$ are closed subsets such that $$X = \bigcup \limits_n S_n$$ then at least one subset $$S_k$$ must contain a non-empty open ball.

a. Let $$B_n$$ denote an open ball of radius $$n$$ about the origin in $$V$$. Apply the Baire theorem to the collection of closed subsets $$\{\overline{\phi(B_n)}\}$$ of $$W$$ to conclude that for some $$k$$, the image $$\phi(B_k)$$ contains a dense subset of some open ball $$B_n(w) \subset W$$ centered at $$w \in W$$.

b. Conclude that for all $$m > 0$$ the set $$\phi(B_{mk})$$ contains a dense set of $$B_{ma}(w)$$. Choosing $$s = mk$$ for suitably large $$m$$ we may assume that $$\phi(B_s)$$ contains $$C_r$$, a ball of radius $$r$$ centered at the origin of $$W$$.

c. Choose $$0 < \delta < 1$$ and $$y \in C_r$$. Show that there exists $$x_1 \in V$$ with $$|x_1| < s$$ such that $$|y - \phi(x_1)| < \delta r$$.

d. Show that there is $$x_2 \in V$$ with $$|x_2| < \delta s$$ and $$|y - \phi(x_1) - \phi(x_2)| < \delta^2 r$$. More generally, for all positive integers $$n$$, there exists $$x_n \in V$$ with $$|x_n| < \delta^{n-1} s$$ and $$|y - \sum \limits_{i=1}^n \phi(x_i)| < \delta^n s$$.

e. Let $$x = \sum \limits_{i=1}^\infty x_i$$. Show that $$x \in B_{\frac{r}{1-\delta}}$$ and that $$\phi(x) = y$$. Conclude that the open mapping theorem is valid.

6. An element $$a$$ of a convex subset $$K$$ of a vector space is called extremal if $$a$$ cannot be written as $$\frac{b+c}{2}$$ for $$b, c \in K$$. 

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a. Suppose $X$ is a compact, Hausdorff space and $B$ denotes the unit ball inside $C(X)$. Show that $f \in B$ is extremal if and only if $|f(x)| = 1$ for all $x \in X$.

b. Show that the smallest convex set containing all extremal points of $B$ is $B$. 