Dyson’s lemma with moving parts

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In its original form [D], Dyson’s Lemma bounds the order of vanishing of a polynomial in two variables at a finite set of points and was used in studying rational approximations to a fixed algebraic irrational number. Bombieri [B] revived interest in Dyson’s Lemma and subsequently Esnault and Viehweg [EV] and Vojta [V1] vastly extended the scope of the result, to polynomials in several variables and to a section of a line bundle on a product of two algebraic curves respectively. The theorems of Vojta and of Esnault and Viehweg are highly refined geometric results which are used to prove the Mordell Conjecture and Roth’s Theorem respectively. The Main Theorems of both [EV] and, up to a factor of two, [V1] were recovered in [N] as special cases of a more general result. Before [N] had been written, Vojta [V3] formulated a result which would vastly improve the main result of [N]. The goal of this note is to prove Vojta’s statement on a product of an arbitrary number of projective lines and to give a counterexample on a product of three or more curves of genus \( \geq 1 \).

In order to state our main result, we need to fix some notation. Let

\[ X = C_1 \times \ldots \times C_m \]

be a product of smooth projective curves defined over an algebraically closed field of characteristic zero with \( g_i \) the genus of \( C_i \). Suppose \( L \) is a line bundle on \( X \) and let \( F_i \) be a fibre of the projection to the \( i^{th} \) factor \( \pi_i : X \to C_i \). Recall the definition ([N] Definition 0.2) of the degree of \( L \) along the \( i^{th} \) factor:

\[ d_i(L) = L \cdot F_1 \cdot \ldots \cdot F_{i-1} \cdot F_{i+1} \cdot \ldots \cdot F_m. \]

For \( 1 \leq i \leq m \) let

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\[\delta_i(L) = \sum_{j=1}^{m} d_j(L).\]

As in [N], since \(L\) will be fixed throughout we will write \(d_i\) in place of \(d_i(L)\)
and similarly for \(\delta_i\). If \(0 \neq s \in H^0(X, L)\) and \(\zeta \in X\) then one can define the index of \(s\) at \(\zeta\), denoted by \(\text{ind}(\zeta, s)\), to be the weighted multiplicity where the \(i^{th}\) factor receives weight \(1/d_i\) (see [V1] Definition 0.2). Finally, let \(I^m\) denote the unit cube in \(\mathbb{R}^m\) and for \(t \in \mathbb{R}_{\geq 0}\) let \(\text{Vol}(t)\) be the euclidean volume of \(\{(x_1, \ldots, x_m) \in I^m : \sum_{i=1}^{m} x_i \leq t\}\). Vojta [V3] raises the following question which would refine [N] Theorem 0.4:

**Question 1.1 (Vojta).** Let \(S = \{\zeta_1, \ldots, \zeta_M\} \subset X\) be a finite subset such that \(\text{card}(S) = \text{card}[\pi_i(S)]\) for all \(i\). Let \(M_i = \max\{2g_i - 2 + M, 0\}\).

Let \(0 \neq s \in H^0(X, L)\) and suppose \(t_j = \text{ind}(\zeta_j, s)\). If \(L' = L(\sum_{i=1}^{m} \delta_i F_i)\), then when is

\[
\left(\prod_{i=1}^{m} d_i\right) \sum_{j=1}^{M} \text{Vol}(t_j) \leq \frac{c_1(L')^{m-1} \cap c_1(L)}{m!}?
\]

Note that an affirmative answer to Question 1.1 improves [N] Theorem 0.4 in two ways, removing the perturbation divisor \(D\) and replacing \(c_1(L')^m\) by \(c_1(L')^{m-1} \cap c_1(L)\). We will show that Question 1.1 has an affirmative answer in the case where \(g_i(C) = 0\) for all \(i\): let \(P = (P^1)^m = P_1^1 \times \ldots \times P_m^1\) and for an \(m\)-tuple of integers \(d = (d_1, \ldots, d_m)\) write

\[\mathcal{C}_P(d) = \bigotimes_{i=1}^{m} \pi_i^* \mathcal{O}_{P_i}(d_i).\]

We will consider in detail the special case of Question 1.1 where \(X \simeq P\) and \(L \simeq \mathcal{C}_P(d)\):

**Theorem 1.2.** With notation as in Question 1.1, let \(\delta = (\delta_1, \ldots, \delta_m)\). Then

\[
\left(\prod_{i=1}^{m} d_i\right) \sum_{j=1}^{M} \text{Vol}(t_j) \leq \frac{c_1(\mathcal{C}_P(d + \delta))^{m-1} \cap c_1(\mathcal{C}_P(d))}{m!}.
\]

Theorem 1.2 is a slight improvement of [EV] Theorem 0.4 where the right hand side has \(c_1(\mathcal{C}_P(d + \delta))^m\) in place of \(c_1(\mathcal{C}_P(d + \delta))^{m-1} \cap c_1(\mathcal{C}_P(d))\). The proof of Theorem 1.2 uses ideas developed by Esnault and Viehweg in [EV] §10; in particular we consider, as promised in [ELN], the moving part of the linear series used to prove [N] Theorem 0.4. As was observed by Esnault and Viehweg [EV],
it is precisely this ‘moving part’ which accounts for the difference of a factor of 2 between [N] Corollary 0.6 and [V1] Theorem 0.4.

We will also show that the answer to Question 1.1 is no for \( m \geq 3 \) if one allows the curves \( C_i \) to have genus \( \geq 1 \). Moreover, we will see that the conjecture fails precisely because the perturbation term \( D \) in [N] Theorem 0.4 is needed in order for this type of Dyson Lemma to hold in full generality. With the perturbation term \( D \) the proof of Theorem 1.2 can be used to refine [N] Theorem 0.4; in other words, Question 1.1 has an affirmative answer for arbitrary \( X \) provided one twists both \( L \) and \( L' \) by \( D \). Interestingly, our counterexample to Question 1.1 uses the line bundles introduced by Vojta in his proof of the Mordell Conjecture [V2, V4].

We begin by recalling the proof of [EV] Theorem 0.4. Let \( \zeta \in \mathbb{P} \) and let \( t \) be a non-negative real number. Define \( \mathcal{J}_{\zeta,d,t} \) to be the ideal sheaf generated by all \( s \in H^0(\mathbb{P}, \mathcal{O}(d)) \) such that \( \text{ind}(\zeta,s) \geq t \). Write
\[
\mathcal{J}_d(s) = \bigcap_{j=1}^M \mathcal{J}_{\zeta_j,d,t_j}.
\]
To alleviate notation in what follows, we will set \( \mathcal{I} = \mathcal{O}(d) \) and \( \mathcal{P} = \mathcal{O}(d+\delta) \). Let \( Z_n \) denote the zero scheme of \( \mathcal{J}_d(s)^n \). The main theorem of [EV] is proven by considering the following exact sequence:
\[
0 \to \mathcal{J}_d(s)^n \to \mathcal{P}^n \to \mathcal{P}^n|_{Z_n} \to 0. \tag{1.3}
\]
One shows by [EV] 5.11 that
\[
h_1(\mathcal{J}_d(s)^n \otimes \mathcal{P}^n) \leq O(n^{m-1})
\]
and then [EV] Theorem 0.4 follows from the long exact cohomology sequence associated to 1.3. For future reference, this exact sequence gives
\[
\lim_{n \to \infty} \frac{h^0(Z_n, \mathcal{P}^n)}{n^m} + \lim_{n \to \infty} \frac{h^0(\mathbb{P}, \mathcal{J}_d(s)^n \otimes \mathcal{P}^n)}{n^m} = \lim_{n \to \infty} \frac{h^0(\mathbb{P}, \mathcal{J}_d(s)^n)}{n^m}. \tag{1.4}
\]
The right hand side of 1.4 can of course be computed by Riemann-Roch:
\[
\lim_{n \to \infty} \frac{h^0(\mathbb{P}, \mathcal{J}_d(s)^n)}{n^m} = \frac{c_1(\mathcal{J})^m}{m!}. \tag{1.5}
\]
The same is true for the second term on the left which measures the moving part of the linear series of sections having index at least \( t_j \) at \( \zeta_j \) for \( 1 \leq j \leq M \). Let \( \tau : Y \to \mathbb{P} \) be a birational map such that \( Y \) is smooth and \( \tau^{-1}\mathcal{J}(s) \) is invertible, say \( \tau^{-1}\mathcal{J}(s) \simeq \mathcal{O}_Y(-E) \) for some Cartier divisor \( E \) on \( Y \). By [N] Theorem 2.1, \( \tau^*\mathcal{J}(s) \) is nef and consequently
\[
\lim_{n \to \infty} \frac{h^0(Y, (\tau^*\mathcal{P}(-E))^n)}{n^m} = \frac{c_1(\tau^*\mathcal{P}(-E))^m}{m!}. \tag{1.6}
\]
But by [EV] 1.9 and 3.5
\[ h^0(Y, \tau^* \mathcal{P}(-E)^\otimes n) = h^0(P, \mathcal{I}_d(s) \otimes \mathcal{P}^\otimes n) \quad \forall n \geq 1; \quad (1.7) \]
hence putting together 1.4, 1.5, 1.6, and 1.7 gives

\[ \lim_{n \to \infty} \frac{h^0(Z_n, \mathcal{P})}{n^m} = \frac{c_1(\mathcal{P})^m}{m!} - \frac{c_1(\tau^* \mathcal{P}(-E))^m}{m!} \quad (1.8) \]

The left hand side of 1.8 can be estimated by 1.7 and [EV] 2.4 and 2.8:

\[ \lim_{n \to \infty} \frac{h^0(Z_n, \mathcal{P} \otimes n)}{n^m} \geq \left( \prod_{i=1}^m d_i \right) \sum_{j=1}^M \text{Vol}(t_j). \quad (1.9) \]

Combining 1.8, and 1.9 gives

\[ \left( \prod_{i=1}^m d_i \right) \sum_{j=1}^M \text{Vol}(t_j) \leq \frac{c_1(\mathcal{P})^m - c_1(\tau^* \mathcal{P}(-E))^m}{m!}. \quad (1.10) \]

Ignoring the \( c_1(\tau^* \mathcal{P}(-E))^m \) term, this is the main theorem of [EV]. The idea of the proof of Theorem 1.2 is first to obtain a lower bound for \( c_1(\tau^* \mathcal{P}(-E))^m \) and second to examine the inequality 1.9 more closely.

**Proof of Theorem 1.2.** To obtain a lower bound for \( c_1(\tau^* \mathcal{P}(-E))^m \), observe first that \( \tau^* \mathcal{L}(-E) \) has a global section, namely the section induced by \( \tau^* s \in H^0(Y, \tau^* \mathcal{L}) \). Using the fact that \( \tau^* \mathcal{P}(-E) \) is nef, we obtain

\[ c_1(\tau^* \mathcal{P}(-E))^m = [c_1(\tau^* \mathcal{P}(-E))^{m-1} \cap c_1(\tau^* \mathcal{L})] + c_1(\tau^* \mathcal{P}(-E))^{m-1} \cap c_1(\tau^* \mathcal{L})(-E)) \geq c_1(\tau^* \mathcal{L})(-E))^m \cap c_1(\tau^* \mathcal{L})(-E)). \quad (1.2.1) \]

To compute the intersection number in 1.2.1, observe that \( c_1(\tau^* \mathcal{L}) - c_1(\tau^* \mathcal{L}) \) is represented by \( \sum_{i=1}^m d_i \tau_i^* F_i \) where \( F_i \) is a fibre of \( \pi_i : P \to P_i \). Fix some \( 1 \leq i \leq m \) and let \( F = F_{\pi_i}^{-1}(\eta) \) for a general point \( \eta \in P_i \). We have the following commutative diagram:

\[
\begin{array}{ccc}
\tilde{Y} & \xrightarrow{j} & Y \\
\downarrow & & \downarrow \\
F & \xrightarrow{f} & P
\end{array}
\]

Here \( \tau_F : \tilde{Y} \to F \) is the blow-up of \( F \) along the inverse image ideal sheaf \( f^{-1}\mathcal{I}_d(s) \). But \( F \) is again a product of projective lines and the inverse image ideal sheaf \( f^{-1}\mathcal{I}_d(s) \) can be computed explicitly by [EV] Lemma 2.9. To express this, note that there is a natural projection \( P : P \to F \) defined by \( (x_1, \ldots, x_m) \mapsto (x_1, \ldots, x_{i-1}, \eta, x_{i+1}, \ldots, x_m) \). Let \( \zeta' = P(\zeta) \) and let \( d' \) be such that \( C_F(d') = C_P(d)[F] \). With this notation,
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\( f^{-1}(\mathcal{I}_d(s)) = \bigcap_{j=1}^{M} \mathcal{I}_{d',j} \).

Observe that since \( F \) is a general fibre of \( \pi_i \), the section \( s \) will not vanish identically along \( F \) and in particular

\[ s | F \in H^0 \left( C_F(d') \otimes \bigcap_{j=1}^{M} \mathcal{I}_{d',j} \right). \]

Write \( s' = s | F \) and write \( \mathcal{I}_d(s) = \bigcap_{j=1}^{M} \mathcal{I}_{d',j} \), so we have \( s' \in H^0 \left( F, C_F(d') \otimes \mathcal{I}_d \right). \)

\( \text{Let } Z'_n = Z \left( \mathcal{I}_d(s' \otimes n) \right). \) Using the exact sequence 1.3 on \( F \) and arguing as in the proof of 1.10 gives

\[ c_1(\tau^* P) m - 1 \cap \tau^* F = c_1(\mathcal{I}) \]

\[ \leq c_1(\mathcal{I}) m - 1 \cap \tau(\mathcal{I}) \]

\[ + \sum_{i=1}^{m} \delta_i \lim_{n \to \infty} \frac{h^0 \left( Z'_n, \mathcal{I} \otimes n \right)}{n^{m-1}}. \]

Equation 1.2.3 holds also with \( Z_n, i \) in place of \( Z_n \) and \( W_n, i, j \) in place of \( W_n, j \). By [EV] 2.4,

\[ \lim_{n \to \infty} \frac{h^0 \left( W_{n,j}, \mathcal{I} \otimes n \right)}{n^m} \geq \left( \prod_{i=1}^{m} d_i \right) \text{Vol}(t_j). \]

We will show that
\[
\lim_{n \to \infty} \frac{h^0 \left( W_n, \mathcal{O}^n \right) - h^0 \left( W_n, \mathcal{L}^n \right) }{n^m} \geq \max_{1 \leq i \leq m} \left\{ \lim_{n \to \infty} \frac{\delta_i h^0 \left( W_n, i, j, \mathcal{P} \otimes n \right) }{nm^{i-1}} \right\}.
\] (1.2.6)

Combining 1.2.4 (for both \( Z_n \) and \( Z_n, i \)), 1.2.5, and 1.2.6 gives
\[
\lim_{n \to \infty} \frac{h^0 \left( Z_n, \mathcal{P} \otimes n \right) }{n^m} \geq \left( \prod_{i=1}^{m} d_i \right) \sum_{j=1}^{M} \operatorname{Vol}(t_j) + \sum_{i=1}^{m} \lim_{n \to \infty} \delta_i h^0 \left( Z_n, i, \mathcal{P} \otimes n \right) .
\] (1.2.7)

Observe that 1.2.7 gives the desired refinement of 1.10: in fact, combining 1.8, 1.2.3, and 1.2.7 gives Theorem 1.2.

It remains to prove 1.2.6. For this we fix some \( 1 \leq i \leq m \) and let \( F = \pi^{-1}(\eta) \) for a general point \( \eta \). Note that
\[
h^0 \left( W_n, \mathcal{O}^n \right) = h^0 \left( F, \mathcal{O}^n \right) - h^0 \left( F, (\mathcal{P} \otimes \mathcal{I}_{\zeta, d, t-1})^n \right) + O(n^{m-2});
\] (1.2.8)
this follows by the same argument used to establish 1.4. Similarly,
\[
h^0 \left( W_n, \mathcal{O}^n \right) = h^0 \left( \mathcal{P}, \mathcal{O}^n \right) - h^0 \left( \mathcal{P}, (\mathcal{P} \otimes \mathcal{I}_{\zeta, d, t})^n \right) + O(n^{m-1}).
\] (1.2.9)

Note that 1.2.9 holds not only for \( \mathcal{P} \) but also for any invertible sheaf \( \mathcal{F} \) on \( \mathcal{P} \) such that \( \mathcal{F} \otimes \mathcal{I}^{-1} \) has a non-zero global section. Fix a point \( \zeta \in \mathcal{P} \) and \( t \in \mathbb{R}^+ \). For \( n \gg 0 \), choose general fibres \( F_1, \ldots, F_{n \delta} \) of \( \pi_i \) and let \( D_n = \sum_{j=1}^{n \delta} F_j \). There is a natural evaluation map
\[
H^0 \left( \mathcal{P}, (\mathcal{P} \otimes \mathcal{I}_{\zeta, d, t})^n \right) \to H^0 \left( D_n, (\mathcal{P} \otimes \mathcal{I}_{\zeta, d, t})^n \right) = \bigoplus_{j=1}^{n \delta} H^0 \left( F_j, (\mathcal{P} \otimes \mathcal{I}_{\zeta, d, t})^n \right).
\]
Since \( H^0(F_j, (\mathcal{P} \otimes \mathcal{I}_{\zeta, d, t})^n) \) can be identified with \( H^0(F, (\mathcal{P} \otimes \mathcal{I}_{\zeta, d, t-1})^n) \) there is a natural map
\[
H^0 \left( \mathcal{P}, (\mathcal{P} \otimes \mathcal{I}_{\zeta, d, t})^n \right) \xrightarrow{\psi} \bigoplus_{n \delta \text{ times}} H^0 \left( F, (\mathcal{P} \otimes \mathcal{I}_{\zeta, d, t-1})^n \right).
\]
Furthermore, since the \( F_j \) are generic, \( \zeta \not\in F_j \) for any \( j \) and consequently
\[
\operatorname{Ker}(\psi_n) = H^0 \left( \mathcal{P}, (\mathcal{P} \otimes \mathcal{I}_{\zeta, d, t})^n(-D_n) \right).
\]
Thus
\[
\begin{align*}
\text{h}^0 (P, (\mathcal{O} \otimes \mathcal{Z}_{d,t}) \otimes n) - \text{h}^0 (P, (\mathcal{O} \otimes \mathcal{Z}_{c,d,t}) \otimes (D_n)) & \\
\leq n \delta \text{h}^0 (F, (\mathcal{O} \otimes \mathcal{Z}_{c,d,t-1}) \otimes n).
\end{align*}
\]

Note that

\[
\text{h}^0 (P, \mathcal{O} \otimes n) - \text{h}^0 (P, \mathcal{O} \otimes n (D_n)) = \text{h}^0 (D_n, \mathcal{O} \otimes n) = n \delta \text{h}^0 (F, \mathcal{O} \otimes n).
\]

Hence, letting \( \zeta = \zeta \) and \( t = t_j \), 1.2.9 applied to both \( \mathcal{O} \) and \( \mathcal{O}(\delta F) \) gives

\[
\begin{align*}
\text{h}^0 (W_{n,j}, \mathcal{O} \otimes n) & - \text{h}^0 (W_{n,j}, \mathcal{O} \otimes n (D_n)) \\
& \geq n \delta \left( (h^0 (F, \mathcal{O} \otimes n)) \right. \\
& - h^0 
\left( F, (\mathcal{O} \otimes \mathcal{Z}_{c,d,t} \otimes (D_n)) \right) \\
& = n \delta \text{h}^0 (W_{n,i+j}, \mathcal{O} \otimes n) + O(n^{-1}),
\end{align*}
\]

where the second step follows from 1.2.8. But \( h^0 (W_{n,j}, \mathcal{O} \otimes n) \geq h^0 (W_{n,j}, \mathcal{O} \otimes n) \) and since 1.2.10 holds for any \( 1 \leq i \leq m \) this concludes the proof of 1.2.6 and hence the proof of Theorem 1.2.

Counterexample to Question 1.1. Thinking about the proof of [N] Theorem 0.4, the potential problem in trying to prove Question 1.1 as stated is the lack of the perturbation term \( D \). The twist by \( D \) was used at a crucial stage in the proof, namely during the inductive step it was used to create a section of \( L \), with suitable indices at a finite set of points, on a proper product subvariety. What could go wrong would be the following: there might exist a proper product subvariety \( Y \subset X \) such that \( L|Y \) has no sections with large index at any point. In fact, Vojta’s work on the Mordell Conjecture [V2, V4] gives a perfect candidate for such a line bundle.

Let \( C \) be a smooth projective curve of genus \( g \geq 1 \) and let \( Y = C \times C \). Let \( F_1 \subset Y \) be a fibre of the first projection, \( F_2 \subset Y \) a fibre of the second projection, and \( \Delta' = \Delta - F_1 - F_2 \). Finally, let \( n = (n_1, n_2, n_3) \) be a 3-tuple of integers. We consider divisors of the form

\[
V_n = n_1 F_1 + n_2 F_2 + n_3 \Delta'.
\]

Since \( F_1 + F_2 \) is ample and \( V_n \cdot (F_1 + F_2) = n_1 + n_2 \), some multiple of \( V_n \) is effective provided \( n_1 + n_2 > 0 \) and \( V_n^2 > 0 \). We have

\[
c_1 (V_n)^2 = 2(n_1 n_2 - gn_3^2).
\]

Hence given any \( \epsilon > 0 \) there exist \( n_1, n_2, n_3 \) such that \( n_1 \gg n_2 \gg 0 \), \( V_n \) is effective, and

\[
c_1 (V_n)^2 \leq \epsilon n_1 n_2. \tag{1.11}
\]

Let \( X = C \times C \times C \). Let \( p : X \to Y \) denote the projection to the first two factors and let \( \pi_3 : X \to C \) denote the projection to the third factor. Also, fix a point \( P \in C \). Let

\[
\mathcal{L} = p^* \mathcal{O}(V_n) \otimes \pi_3^* \mathcal{O}_C (P).
\]
Choose \(0 \neq s \in H^0(Y, \mathcal{O}_Y(V_n))\) and let \(t \in H^0(C, \mathcal{O}_C(P))\) be the section whose divisor of zeroes is \(P\). Then \(\text{ind}(\zeta, p^*s \otimes \pi_3^*t) \geq 1\) for any point \(\zeta \in X\) with \(\pi_3(\zeta) = P\). To verify that this contradicts Question 1.1, note that \(d(Z) = (n_1, n_2, 1)\) and that \(\text{Vol}(1) = 1/6\). But if \(1/n_2 \leq \delta\) and \(n_2/n_1 \leq \delta\) an easy computation using 1.11 shows that

\[
c_1\left(\mathcal{Z}(\delta_1 p^*F_1 + \delta_2 p^*F_2)\right)^2 \cap c_1(Z) \leq (3\epsilon + 10\delta)n_1n_2.
\]

(1.12)

For \(\delta\) and \(\epsilon\) sufficiently small, 1.12 contradicts Question 1.1.

It is clear from the point of view of [N] what is going wrong in this counterexample. At the inductive step of the argument, one will need to restrict precisely to \(C \times C \times P\) and produce a section of \(Z\) with index close to 1 at \(p(\zeta)\). Such a section does not exist, however, by Vojta’s version of Dyson’s Lemma [V1]. Thus one needs to twist \(Z\) by something in order to make the argument work. In [N] I chose the simplest possible choice of twist guaranteed to make the inductive argument work; in practice, one can no doubt often get away with less.

References

[V3] P. Vojta: Dyson’s Lemma in general except that \(m = 0\); Rough Notes, unpublished manuscript, 1990