SESHADRI CONSTANTS ON ABELIAN VARIETIES

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Abstract. We show that on a complex abelian variety of dimension two or greater the Seshadri constant of an ample line bundle is at least one. Moreover, the Seshadri constant is equal to one if and only if the polarized abelian variety splits as a product of a principally polarized elliptic curve and a polarized abelian subvariety of codimension one. We also examine the case when the Seshadri constant is not one and obtain lower bounds when the dimension of the abelian variety is small.

1. Introduction. In [EL], Ein and Lazarsfeld, following Demailly [D], define various Seshadri constants on algebraic varieties. Seshadri constants provide a local measure of positivity for ample line bundles and have received attention recently as they provide a method for producing sections of adjoint bundles (cf. [EL] proposition 3.4). Of interest to us here will be the global Seshadri constant on an abelian variety. Let us recall the definition. Suppose that $X$ is a smooth complex projective variety and $L$ an ample line bundle on $X$. For $x \in X$, the local Seshadri constant of $L$ at $x$ is defined by

$$\epsilon(x, L) = \inf_{C \ni x} \frac{L.C}{\text{mult}_x (C)};$$

here the infimum runs over all integral curves $C \subset X$ passing through $x$. One can then define the global Seshadri constant

$$\epsilon(X) = \inf_{x,L} \epsilon(x, L),$$

where the infimum ranges over all $x \in X$ and all ample line bundles $L$.

Suppose $A$ is a complex abelian variety of dimension $g$. Since $A$ acts on itself transitively by translation and since translation preserves numerical equivalence, $\epsilon(x, L)$ does not depend on $x$ and so we will write $\epsilon(L)$ for the Seshadri constant associated to an ample line bundle on $A$. In [EKL], Ein, Küchle, and Lazarsfeld show that for any ample $L$ on a smooth projective variety $X$, $\epsilon(\eta, X) \geq 1/\dim X$ for a very general point $\eta \in X$. It follows that on an abelian variety we always have $\epsilon(L) \geq 1/g$ for any ample line bundle $L$ on $A$. In fact, it was shown in [Nak1] that $\epsilon(L) \geq 1$. The goal of this note is to strengthen these results by showing
THEOREM 1.1. Let A be an abelian variety of dimension \( g \geq 2 \). Then \( \epsilon(L) \geq 1 \) for all ample line bundles L. Moreover, there exists an ample L with \( \epsilon(L) = 1 \) if and only if \( A \cong E \times B \) where E is an elliptic curve and B is an abelian variety of dimension \( g - 1 \).

The global Seshadri constant \( \epsilon(A) \) seems more difficult to control. We will give some partial results in this direction but the only case which can be dealt with satisfactorily using the techniques of the present paper appears to be that of an abelian surface:

THEOREM 1.2. Let A be an abelian surface which is not isomorphic to a product of two elliptic curves. Then

\[
\epsilon(A) \geq 4/3.
\]

Equality can hold in Theorem 1.2 as is shown by an example of Steffens [S]. In fact, Steffens shows that on a general abelian surface \( A \), \( \epsilon(A) = 4/3 \). In the higher dimensional case, we are only able to show

THEOREM 1.3. Let A be a simple abelian variety of dimension \( g \geq 2 \). Then \( \epsilon(A) > 1 \).

Unfortunately, the method of proof of Theorem 1.3 is not effective. We present an alternative proof for the case where \( 3 \leq g \leq 6 \) which can be used to give an effective lower bound but since it is presumably far from optimal we did not make the computations.

There are many interesting questions left unanswered here. For example, if \( A \) is a general abelian variety of dimension \( g \) then the Picard number is one (hence \( A \) is simple) and so it follows from Theorem 1.3 that \( \epsilon(A) > 1 \). One might ask how this number behaves as \( g \to \infty \), i.e., is it bounded or does it grow with \( g \)? A simpler and apparently even more mysterious question would be what, if any, global geometric information is encoded in the constant \( \epsilon(A) \)? Finally, if \( L \) is an ample line bundle with \( \mathcal{O}_A(L) \cong \mathcal{O}_A(D) \) for an effective divisor \( D \), then the singularities of \( D \) give an additional tool in investigating the Seshadri constant of \( L \) (cf. for example Lemma 2.3 below). It might be interesting to try to use this technique in studying the reducibility questions raised in [Nak1].

Our proofs of Theorems 1.1, 1.2, and 1.3 use the techniques of zero-estimates as developed in [Nak1, Nak2, ELN]. In fact, the main result of [Nak1, SV] can be recovered quite quickly from Theorem 1.1 (this is done at the end of §2) and it was this problem that originally hinted at the possible global geometric significance of Seshadri constants on abelian varieties.

Notation and conventions.

- We will take derivatives of sections of line bundles as in [Nak1] and [ELN]. Let \( L \) be a line bundle on an abelian variety \( A \) and \( \sigma \in H^0(A, L) \) a global section. Let \( v \in T_0(A) \) be a nonzero tangent vector and \( F \) the corresponding
translation invariant vector field. Then we can differentiate $\sigma$ along $F$ to obtain $D_F(\sigma) \in H^0(Z(\sigma), L)$ where $Z(\sigma)$ denotes the zero scheme of $\sigma$ (cf. [Nak1] §1 for details). If $D = Z(\sigma)$ we will write $D_F(\sigma)$ as $\partial_D|D$ or simply as $\partial(D)|D$.

- Given a point $x \in A$ and a subvariety $V \subset A$, let

$$t_V(x) = \{x + v : v \in V\}.$$

- If $V \subset A$ is a subvariety and $P \in V$ a point, let $T_P(V) \subset T_P(A) \cong T_0(A)$ denote the Zariski tangent space; here the identification of $T_P(A)$ with $T_0(A)$ is made by translating $P$ to the origin. We will also write $C_P(V)$ for the tangent cone of $V$ at $P$. If $\pi : X \to A$ denotes the blow-up of $A$ at $P$ with exceptional divisor $E \cong \mathbb{P}^{s-1}$ then there is a natural embedding $C_P(V) \subset \mathbb{P}^{s-1}$ and we will accordingly view the tangent cone as a subscheme of projective space.

- If $P \in A$ is a point and $\beta \in \mathbb{R}^+$ then we work with “symbolic powers” $\mathcal{I}_P^{(\beta n)}$ defined by

$$H^0\left(A, L \otimes \mathcal{I}_P^{(\beta n)}\right) = \left\{\sigma \in H^0(A, L) | \text{mult}_P(\sigma) \geq n\beta\right\}.$$

- We denote the base locus of the linear series $|nL \otimes \mathcal{I}_P^{(\beta n)}|$ by $\text{BS}|nL \otimes \mathcal{I}_P^{(\beta n)}|$.

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2. An auxiliary result. Throughout this section $A$ will denote a complex abelian variety of dimension $\geq 2$. Note that by [Nak1] lemma 2.3, we know that $\epsilon(A) \geq 1$. There is a slight difficulty in the proof of Theorem 1.1 stemming from the fact that $\epsilon(L)$ is defined as an infimum so it is not a priori clear that this infimum is actually achieved by some curve. So we will first deal with the extremal case where the infimum is achieved and then Theorems 1.1 and 1.2 will quickly follow (Theorem 1.2 is actually simpler and follows readily from (2.2.1) and (2.2.2) below):

**Theorem 2.1.** Suppose that there is an ample line bundle $L$, an integral curve $C \subset A$, and a point $P \in C$ such that

$$\text{mult}_P(C) = L.C.$$

Then $L.C = 1$. 
By applying a translation, we can assume that $P = 0$, the origin in $A$. The proof of Theorem 2.1 will be by induction on $g$. We make use of the induction on dimension as follows. Suppose that $C$ is degenerate, i.e., there exists a proper abelian subvariety of $A$ containing $C$. Let $B$ denote the abelian subvariety generated by $C$. If $\dim B = 1$ then of course $C$ is an elliptic curve and we are done because in this case $\text{mult}_P(C) = 1$ and so (2.1.1) implies that $L.C = 1$. If $\dim B \geq 2$ then we can restrict $L$ to $B$ and use induction on dimension to conclude that $L.C = 1$. We will treat the cases $\dim B = 2$ and $\dim B \geq 2$ separately, first giving the proof of Theorem 2.1 for an abelian surface where the argument is particularly simple and transparent:

**Lemma 2.2.** Let $A$ be an abelian surface. Suppose $C \subset A$ is an irreducible curve, $x \in C$, and $m = \text{mult}_x(C)$. Suppose there exists an ample line bundle $L$ with $L.C = m$. Then $L.C = 1$.

**Proof of Lemma 2.2.** By the Hodge index theorem

\begin{equation}
L.C \geq \sqrt{L^2 - C^2}.
\end{equation}

If $C^2 = 0$ then $C$ is degenerate, hence a smooth elliptic curve and we conclude that $L.C = 1$. Thus we may assume that $C^2 > 0$. The Riemann-Roch theorem on $A$ implies that $L^2 \geq 2$ and we will show that

\begin{equation}
C^2 \geq m(m - 1) + 2.
\end{equation}

A quick arithmetic check shows that (2.1.1), (2.2.1), and (2.2.2) are compatible only when $m = 1$.

To prove (2.2.2), let $\phi : C \to \mathbb{P}^1$ denote the Gauss map of the divisor $C$ which is defined on the set of smooth points of $C$ as the projectivization of the map $\psi(P) = T_P(C) \subset T_0(A)$. Since $C^2 > 0$ by assumption, $C$ is ample and [BL] proposition 4.2 implies that $\phi$ is dominant. Since $g(C) \geq 1$ the degree of $\phi$ must be at least 2. For a general first order derivative $\partial$, $\partial(C)|C$ will vanish at all singular points of $C$ and also at $\deg(\phi)$ smooth points. The order of vanishing of $\partial(C)$ at $x$ is at least $m - 1$ and $\partial(C)|C$ is a section of $O_C(C)$; consequently we conclude that

\begin{equation}
C.C \geq m(m - 1) + 2
\end{equation}

as desired.

Thus for the remainder of the proof of Theorem 2.1, we may assume that $C$ is nondegenerate and $\dim A \geq 3$. Choose an effective divisor $D$ so that $O_A(L) \simeq O_A(D)$. Choosing $D$ general, we may assume that $D$ is reduced ([BL] chapter 4, proposition 1.7). The hypothesis that $C$ is nondegenerate will be used in an essential way in the following two lemmas:
Lemma 2.3. Suppose there exists a nondegenerate curve $C \subset A$ and an ample line bundle $L$ satisfying (2.1.1). Suppose moreover that $\mathcal{O}_A(L) \simeq \mathcal{O}_A(D)$ for a reduced divisor $D$. Then $D$ is smooth and the Gauss map

$$\phi: D \to \mathbb{P}^{g-1}$$

is a finite morphism.

Proof of Lemma 2.3. We will show that if $x \in \text{Sing}(D)$ then $x+C \subset \text{Sing}(D)$. If $\text{Sing}(D)$ is nonempty, it follows that $C+\text{Sing}(D) = \text{Sing}(D)$ and hence $C+W = W$ for each irreducible component $W$ of $\text{Sing}(D)$. Consequently

$$\text{Stab}(W) = \{x \in A: x+W = W\}$$

is a positive dimensional proper subgroup variety and $C \subset \text{Stab}(W)$. But this violates the hypothesis that $C$ is nondegenerate and so we can conclude as desired that $\text{Sing}(D)$ is empty.

Let

$$Y = \overbrace{C \times \cdots \times C}^{g \text{ times}}$$

and let $\psi_g: Y \to A$ be the addition map defined by $\psi_g(P_1, \ldots, P_g) = P_1 + \cdots + P_g$. Since $C$ is nondegenerate, $\psi_g$ is a surjective, generically-finite morphism. Choose smooth points $P_1, \ldots, P_g$ so that the map $\psi_g$ is smooth at $P = (P_1, \ldots, P_g)$. It follows that the induced map on tangent spaces

$$(2.3.1) \hspace{1cm} d\psi_g: T_P(Y) \to T_{\psi_g(P)}(A)$$

is an isomorphism. We have a natural decomposition $T_P(Y) = \bigoplus_{i=1}^g T_{P_i}(C)$. Moreover by translating to the origin, we can identify the tangent spaces $T_{P_i}(C)$ and $T_{\psi(P)}(A)$ with subspaces of $T_0(A)$. With these identifications, the map $d\psi_g$ is just the addition map and hence (2.3.1) implies

$$(2.3.2) \hspace{1cm} T_{P_1}(C) + \cdots + T_{P_g}(C) = T_0(A).$$

Now we wish to show that if $x \in \text{Sing}(D)$ then $x+C \subset \text{Sing}(D)$. First note that (2.1.1) implies that $x+C \subset D$ since otherwise we would have $D.C \geq 2 \text{ mult}_i (x+C)$ by [Fu] theorem 12.4. Choose nonzero derivations $\partial_1, \ldots, \partial_g$ in the directions of $T_{P_1}(C), \ldots, T_{P_g}(C)$. Consider the sections $\partial_i(D)|C$. We know that $\partial_i(D)$ vanishes at $x$ for all $i$ since $x \in \text{Sing}(D)$. But $\partial_i(D)$ also vanishes at $P_i$ by choice of $\partial_i$ and the fact that $x+C \subset D$. Therefore, if $\partial_i(D)|C$ is not identically zero then $D.C > \text{ mult}_i (x+C)$ contradicting (2.1.1). Thus $\partial_i(D)|C = 0$ for all $i$ and since, by 2.3.2, the derivations $\partial_i$ span $T_0(A)$ it follows that $x+C \subset \text{Sing}(D)$. This shows, as argued above, that $\text{Sing}(D)$ must be empty and hence that $\phi$ is
a morphism; \( \phi \) is finite because it is defined by sections of \( \mathcal{O}_D(D) \) which is an ample invertible sheaf.

**Lemma 2.4.** Choose a general point \( \eta \in C \) and let

\[ V = \text{supp}(D \cap t_{-\eta}(D)). \]

The hypothesis (2.1.1) implies that for each irreducible component \( Y \subset V \)

\[ D = Y + C = \{ y + x \mid y \in Y \text{ and } x \in C \}. \]

**Proof of Lemma 2.4.** Suppose \( x \in V \) and consider

\[ t_x(C) = x + C = \{ x + y \mid y \in C \}. \]

By (2.1.1) it follows that

\[ \text{mult}_x(t_x(C)) = D.C. \tag{2.4.1} \]

On the other hand, since \( x \in V \) it follows that \( x \in D \) and \( x + \eta \in D \). Thus, if \( D \)
meets \( t_x(C) \) properly, then

\[ D.C = D.t_x(C) \geq \text{mult}_x(t_x(C)) + \text{mult}_{x+\eta}(t_x(C)) \]

and, by 2.4.1, this contradicts (2.1.1). We conclude that \( t_x(C) \subset D \) for all \( x \in V \)
and consequently \( V + C \subset D \). Hence for each irreducible component \( Y \subset V \) we have

\[ Y \subset Y + C \subset D. \tag{2.4.2} \]

We saw in the proof of Lemma 2.3 that \( D \) is smooth. Since \( D \) is also ample, this
implies that \( D \) must be irreducible. Thus (2.4.2) implies that either \( C + Y = Y \) or
\( C + Y = D \). Since \( C \) is nondegenerate, we see as in the proof of Lemma 2.3 that
\( C + Y \neq Y \). Thus \( C + Y = D \) and this concludes the proof of Lemma 2.4.

**Proof of Theorem 2.1.** As above, choose \( D \) reduced so that \( \mathcal{O}_A(L) \simeq \mathcal{O}_A(D) \).
By Lemma 2.3, the Gauss map

\[ \phi: D \to \mathbb{P}^{g-1} \]

is a finite morphism. Suppose that \( \text{mult}_P(C) \geq 2 \), i.e., suppose that \( C \) is singular at
\( P \). Then \( \dim T_P(C) \geq 2 \) so we can choose two independent derivations \( \partial_1, \partial_2 \in T_P(C) \subset T_0(A) \). Consider \( \partial_1(D)|D \) and \( \partial_2(D)|D \) which we can identify with \( \phi^*(H_1) \) and \( \phi^*(H_2) \) for the hyperplanes in \( \mathbb{P}^{g-1} \) corresponding to \( \partial_1 \) and \( \partial_2 \). By
Lemma 2.4. \( D = Y + C \) for any irreducible component \( Y \subset V \); in particular \( v + C \subset D \) for all \( v \in Y \). But by the choice of \( \partial_1 \) and \( \partial_2 \) it follows that \( \partial_1(D) \) and \( \partial_2(D) \) vanish at \( v \) for all \( v \in Y \). Thus \( Y \subset \phi^{-1}(H_1 \cap H_2) \) and it follows that \( Y \) is contracted by \( \phi \) (here we use the assumption that \( \dim A \geq 3 \) so that \( \dim Y \geq 1 \)). This is impossible, however, since \( \phi \) is finite. Thus it must be that \( \text{mult}_P(C) = 1 \) and so by (2.1.1) \( D.C = 1 \). This concludes the proof of Theorem 2.1.

**Corollary 2.5.** With hypotheses as in Theorem 2.1, \( A \) decomposes as a product of an elliptic curve and an abelian subvariety of codimension 1.

**Proof of Corollary 2.5.** Since \( L.C = 1 \) it follows that \( C \) is an elliptic curve. To finish the proof, we apply:

**Lemma 2.6.** Suppose \( L \) is an ample line bundle on an abelian variety \( A \) and \( E \subset A \) is an elliptic curve with \( L.E = 1 \). Then there exists an abelian subvariety \( B \subset A \) of codimension 1 such that \( A \simeq E \times B \). Moreover, if \( \pi_1: E \times B \to E \) and \( \pi_2: E \times B \to B \) denote the projections to the first and second factors respectively, then there exists a point \( P \in E \) and a divisor \( D \in B \) such that

\[
\mathcal{O}_A(L) \simeq \mathcal{O}_{E \times B}(\pi_1^*P + \pi_2^*D)
\]

under the identification of \( A \) with \( E \times B \).

**Proof of Lemma 2.6.** Let \( A/E \) be the quotient of \( A \) by \( E \) and consider the natural projection morphism

\[
\pi: A \to A/E.
\]

Write \( \mathcal{O}_A(L) \simeq \mathcal{O}_A(D) \) for some effective divisor \( D \). Since \( D.E = 1 \) it follows that there is an irreducible component \( Y \) of \( D \) which maps generically 1-1 onto \( A/E \). Thus there is a birational map \( \phi: A/E \dashrightarrow Y \subset A \). It follows that \( Y \) is (a translate of) an abelian subvariety of \( A \) isomorphic to \( A/E \); we can then take \( B \) to be a suitable translate of \( Y \). The isomorphism \( E \times B \simeq A \) is given by the addition map. The second part of Lemma 2.6 follows from the fact that \( \mathcal{O}_A(L) \simeq \mathcal{O}_A(Y + Z) \) where \( \pi(Z) \neq A/E \); under the identification with \( E \times B \), \( Y = \pi_1^*P \) for some point \( P \in E \) and similarly \( Z = \pi_2^*D \) where \( D \) is the image of \( \pi(Z) \) under the natural isomorphism between \( A/E \) and \( B \).

We conclude this section by showing how Lemma 2.6 and Theorem 2.1 can be used to give a quick proof of the main result in [Nak1, SV], namely that if \( (A, \Theta) \) is a principally polarized abelian variety of dimension \( g \) such that \( \text{mult}_P(\Theta) = g \) for some point \( P \in A \), then \( A \) is a direct product of \( g \) elliptic curves. Let \( S_{g-1} \) denote the \( g-1 \)-fold singular locus of \( \Theta \). One shows (cf. [Nak1] lemma 1.5) that for any curve \( C \subset S_{g-1} \), \( \text{mult}_P(C) = \Theta.C \). Applying Theorem 2.1 and Lemma 2.6.
forces \((A, \Theta)\) to split into a product of an elliptic curve and an abelian subvariety of codimension one to which one can apply induction on dimension to conclude the argument.

3. Proofs of theorems.

Proof of Theorem 1.1. The first statement in Theorem 1.1 is [Nak1] lemma 2.3. One direction of the second assertion is trivial, namely that if \(A \simeq E \times B\) then there is a line bundle \(L\) with \(\epsilon(L) = 1\). For let \(\pi_1: A \to E\) and \(\pi_2: A \to B\) be the two projections and choose points \(P \in E\) and \(Q \in B\). Also let \(C_Q = C \times Q \subset A\). Given any ample divisor \(D\) on \(B\), \(L = \mathcal{O}_A(\pi_1^*(P) + \pi_2^*(D))\) is an ample line bundle on \(A\) with \(\mathcal{O}_A(C_Q) = \mathcal{O}_A(D)\).

For the other direction of Theorem 1.1, suppose \(A\) is an abelian variety and \(L\) an ample line bundle on \(A\) with \(\epsilon(L) = 1\). By a theorem of Campana and Peternell [CP], there exists a proper subvariety \(V \subset X\) of dimension \(\geq 1\) and a point \(P \in V\) such that

\[
\text{mult}_P (V) = \deg_L(V),
\]

where the multiplicity is defined to be the degree of the tangent cone \(C_P(V)\). If \(V\) happens to be a curve, then Theorem 1.1 follows immediately from Theorem 2.1 and Corollary 2.5. When \(V\) is not a curve, one can reduce to the case of a curve by cutting \(V\) down with appropriate divisors. More precisely, choose a reduced effective divisor \(D\) with \(\mathcal{O}_A(L) \simeq \mathcal{O}_A(D)\). Let \(r = \dim V\) and let \(D_1, \ldots, D_r\) be general translates of \(D\) through \(P\). Since the intersection of all translates of \(D\) through \(P\) is finite (it is a translate of the stabilizer of \(D\)), \(V \cap D_1 \cap \cdots \cap D_r\) is a proper intersection. By (3.1) and [Fu] theorem 12.4 it follows that

\[
\begin{align*}
(\text{a}) & \quad C_P(V) \cap C_P(D_{i_1}) \cap \cdots \cap C_P(D_{i_r}) \text{ is empty,} \\
(\text{b}) & \quad V \cap D_{i_1} \cap \cdots \cap D_{i_r} \text{ is supported entirely on the point } P, \text{ and} \\
(\text{c}) & \quad \text{mult}_P (D_i) = 1 \text{ for all } 1 \leq i \leq r.
\end{align*}
\]

Let \(C\) be an irreducible component of \(V \cap D_1 \cap \cdots \cap D_{r-1}\). Then (a) and (b) imply that \(C_P(D_r) \cap C_P(C)\) is empty and \(P\) is the only point of intersection of \(C\) and \(D_r\). Hence applying [Fu] theorem 12.4 one more time and using (c) gives

\[
L.C = D_r.C = \text{mult}_P(C) \cdot \text{mult}_P(D_r) = \text{mult}_P(C)
\]

and we are reduced to Theorem 2.1 and Corollary 2.5.

Proof of Theorem 1.2. Theorem 1.2 is a direct consequence of (2.2.1) and (2.2.2) of the previous section. One can check also that the extremal case \(L.C = 4/3 \text{ mult}_P(C)\) can occur only when

\[
(\text{i}) \quad C\text{ is numerically equivalent to a multiple of } L.
\]
(ii) \( \text{mult}_P(C) = 3 \) and \( L.C = 4 \) or \( \text{mult}_P(C) = 6 \) and \( L.C = 8 \), and

(iii) the Gauss map of \( C \) has degree 2.

This gives some indication that the example of Steffens [S] is essentially the only example where \( \epsilon(L) = 4/3 \).

Proof of Theorem 1.3. Since \( A \) is simple, it contains no elliptic curves and so by Theorem 1.1, \( \epsilon(L) > 1 \) for all ample \( L \) on \( A \). Unfortunately there is nothing which a priori eliminates the possibility that there might be a sequence of line bundles \( L_i \) whose Seshadri constants approach 1. We will use the techniques developed in [ELN] §3 to show

PROPOSITION 3.2. Let \( A \) be a simple abelian variety of dimension \( g \geq 2 \). Then

\[
\epsilon(A) \geq \frac{\sqrt[2]{c_1(L)^g}}{g}.
\]

By Proposition 3.2, there exists \( \delta > 0 \) such that \( \epsilon(L) \geq 1 + \delta \) whenever \( c_1(L)^g > g^g \). Thus to prove Theorem 1.3 it suffices to show that there are only finitely many possible Seshadri constants associated to ample line bundles \( L \) with \( c_1(L)^g \leq g^g \). But by a result of Narasimhan and Nori ([NN] theorem 1.1), there are only finitely many polarizations of \( A \), up to isomorphism, with bounded degree. Since Seshadri constants are invariant under an automorphism of \( A \), this shows that for \( \delta \) sufficiently small, \( \epsilon(L) \geq 1 + \delta \) for all \( L \) with \( c_1(L)^g \leq g^g \) and this concludes the proof of Theorem 1.3.

The proof of Proposition 3.2 uses an idea of Nadel [Nad] which appears in a very similar context in work of Ein, Küchle, and Lazarsfeld [EKL]. Proposition 3.2 is a direct corollary of the following result:

LEMMA 3.3. Let \( L \) be an ample line bundle on an abelian variety and let \( C \) be a nondegenerate curve. Then for any point \( P \in C \)

\[
\frac{L.C}{\text{mult}_P(C)} \geq \frac{\sqrt[2]{c_1(L)^g}}{g}.
\]

By [Nak1] lemma 2.3, Lemma 3.3 holds if \( \frac{\sqrt[2]{c_1(L)^g}}{g} \leq 1 \) so we can assume without loss of generality that

\[
\frac{\sqrt[2]{c_1(L)^g}}{g} > 1.
\]

Suppose there exists a curve \( C \) with

\[
\frac{L.C}{\text{mult}_P(C)} < \frac{\sqrt[2]{c_1(L)^g}}{g}.
\]
Choose $\delta > 0$ satisfying

$$1 < \frac{L.C}{\operatorname{mult}_P(C)} + \delta < \frac{\sqrt{c_1(L)^3}}{g}$$

and let

$$\alpha = \frac{L.C}{\operatorname{mult}_P(C)} + \delta.$$  

For $\beta \in \mathbb{R}^+$ consider the graded linear series

$$R_\beta = \bigoplus_{n=0}^{\infty} R_\beta(n) = \bigoplus_{n=0}^{\infty} H^0\left( nL \otimes \mathcal{I}_P(n) \right).$$

Since $g\alpha < \sqrt{c_1(L)^3}$, the Riemann-Roch theorem implies that $R_\beta(n)$ is nonempty for $n \gg 0$ whenever $\beta \leq g\alpha$. Therefore the results of [ELN] §3 apply. In particular for any $\beta \leq g\alpha$ and any $n \gg 0$ we can associate a subvariety $Z_\sigma(R_\beta(n))$ defined by

$$Z_\sigma(R_\beta(n)) = \{ x \in A | \operatorname{mult}_x(D_n) \geq n\sigma \text{ for all } D_n \in R_\beta(n) \}.$$  

By [ELN] lemma 3.8, the subvarieties $Z_\sigma(R_\beta(n))$ stabilize as $n \to \infty$ and we denote the limit by $Z_\sigma(\beta)$. Recall that one defines the index of a section $\sigma \in H^0(A, nL)$ at a point $P \in A$ to be $\operatorname{mult}_P(\sigma)/n$. With this notation, the set $Z_\sigma(R_\beta)$ admits the following simple description: imposing index $\beta$ at the point $P$ forces vanishing at other points and $Z_\sigma(R_\beta)$ is simply the set of points where one is forced to have index at least $\sigma$. The proof of Lemma 3.3 will follow from the following lemma about graded linear series on an abelian variety:

**L E M M A 3.4.** Let $A$ be an abelian variety, $P \in A$ a point and $R_\beta$ the associated graded linear series. Then for all $\gamma \geq 0$

$$Z_\sigma(R_\beta) \subset Z_{\gamma+\sigma}(R_{\gamma+\beta}).$$

**Proof of Lemma 3.4.** The idea is simply to differentiate sections of $R_{\gamma+\beta}(n)$. Suppose $s \in R_{\gamma+\beta}(n)$ and suppose $D$ is a differential operator of weight $\leq n\gamma$. Suppose for the moment that $D(s)$ is globally defined as a section of $L^{\otimes n}$. Since $s \in R_{\gamma+\beta}(n)$, $\operatorname{mult}_P(s) \geq n(\gamma+\beta)$ and so $\operatorname{mult}_P(D(s)) \geq \beta n$. Hence, by definition, $D(s) \in R_\beta(n)$ and so $s$ must have index at least $\sigma$ along $Z_\sigma(R_\beta)$. But $D$ was an arbitrary differential operator of weight $\leq n\gamma$ and we conclude that $s$ has index at least $\gamma + \sigma$ along $Z_\sigma(R_\beta)$, as desired. To make the proof rigorous, one needs to define $D(s)$ globally as in [ELN, Nak1]; this can be done if we twist $L$ by an arbitrarily small $\mathbb{Q}$-multiple $\epsilon L$. Taking the limit as $\epsilon \to 0$ then establishes Lemma 3.4.
Proof of Lemma 3.3. Consider the filtration
\[ R_{g\alpha} \subset R_{(g-1)\alpha} \subset \cdots \subset R_{\alpha}. \]
We will show inductively that for \( \epsilon \) sufficiently small, namely for \( \epsilon < \delta \),
\[ Z_i(R_{g\alpha}) \supset C + \cdots + C. \]
(3.3.1)
Applying (3.3.1) when \( i = g \) shows that \( C \) must be degenerate (since \( R_{g\alpha} \neq 0 \)) which contradicts the hypothesis in Lemma 3.3.
Let \( 0 \neq D_i \in R_{\alpha}(n) \) for \( n \gg 0 \). Consider the case \( i = 1 \) of (3.3.1). We claim that \( C \subset D_1 \); if not, then
\[ nL.C = D_1.C \geq \text{mult}_P(D_1) \cdot \text{mult}_P(C) \geq \alpha n \cdot \text{mult}_P(C) \]
which contradicts the definition of \( \alpha \). The same argument can be applied to derivatives of \( D_1 \) and will continue to yield the same contradiction until the index of the derivative exceeds \( \delta \). Hence whenever \( \epsilon < \delta \)
\[ C \subset Z_\epsilon(R_{g\alpha}). \]
By Lemma 3.4 this implies that
\[ C \subset Z_{\epsilon+\alpha}(R_{2\alpha}). \]
Thus \( \text{mult}_C(D_2) \geq n(\epsilon + \alpha) \) and, by the argument given in (3.3.2), \( D_2 \) must contain every translate \( x + C \) for \( x \in C \), i.e. \( C + C \subset D_2 \). A refinement gives \( C + C \subset Z_\alpha(R_{2\alpha}). \) Iterating this argument yields (3.3.1) and concludes the proof of Lemma 3.3.

We conclude this note with an effective proof of Theorem 1.3 for \( 3 \leq g \leq 6 \). This proof uses the same techniques as in the proof of Lemma 3.3 and also the Product Theorem of Faltings [F]. Suppose \( A \) is a simple abelian variety of dimension \( \geq 2 \) and \( \epsilon(A) = 1 \). By Theorem 1.1 this is only possible if there is a sequence \( L_i \) of line bundles with \( \epsilon(L_i) > 1 \) for all \( i \) and \( \lim_{i \to \infty} \epsilon(L_i) = 1 \). In particular, there must exist curves \( C_i \) and points \( P_i \in C_i \) with \( \text{mult}_{P_i}(C_i) < L_i.C_i \) for all \( i \) and
\[ \lim_{i \to \infty} \frac{L_i.C_i}{\text{mult}_{P_i}(C_i)} = 1. \]
This implies that \( \lim_{i \to \infty} \text{mult}_{P_i}(C_i) = \infty \) and also
\[ \lim_{i \to \infty} \deg_{L_i}(C_i) = \infty. \]
(3.5)
Write
\[ R_\beta(L_i) = \bigoplus_{i=0}^{\infty} H^0 \left( n L_i \otimes T_{P_i}^{(n \beta)} \right). \]

For any ample line bundle \( L \) and point \( P \in A \), we define a constant \( \kappa(L, P) \) which measures how large the index of a section of \( nL \) can be at \( P \) as \( n \to \infty \):
\[ \kappa(L, P) = \sup \{ \alpha | R_\alpha(L) \neq 0 \}. \]

It is not difficult to check, using translations, that \( \kappa(L, P) \) does not depend on \( P \) and we will accordingly write it as \( \kappa(L) \). The effective proof of Theorem 1.3 consists in obtaining upper and lower bounds for \( \kappa(L) \) as \( i \to \infty \). The upper bound is obtained as in the proof of Lemma 3.3 by showing that \( A \subset Z_\sigma(R_\beta(L_i)) \) for suitable \( \sigma \) and \( \beta \). The lower bound is obtained by carefully counting the number of conditions required to impose large multiplicity at the point \( P_i \) in question.

We first obtain the upper bound on the \( \kappa(L_i) \). To alleviate the notation, fix some \( i \gg 0 \) and write \( L = L_i \), \( C = C_i \), \( P = P_i \), and \( R_\beta = R_\beta(L_i) \). Choose some \( \delta \in \mathbb{Q} \) satisfying \( 0 < \delta \ll 1 \). Since \( i \) is large we may assume that
\[ \frac{L.C}{\text{mult}_P(C)} < 1 + \delta. \]

As in (3.3.2) above, (3.6) implies that
\[ C \subset BS \left| n L \otimes T_{P}^{(n(1+\delta))} \right| \text{ for all } n > 0. \]

Lemma 3.4 then implies that \( C \subset Z_5(R_{1+2\delta}) \) and also that \( C \subset Z_6(R_{1+3\delta}) \). Suppose that \( C \) is an irreducible component of both \( Z_5(R_{1+2\delta}) \) and \( Z_6(R_{1+3\delta}) \). Then the ‘one factor’ product theorem, [ELN] theorem 2.1, implies that
\[ \deg_L(C) \leq \frac{c_1(L)^g}{\delta^{g-1}}. \]

By Proposition 3.2, we may assume that \( c_1(L)^g \) is bounded above (by, say, \( g^8 + 1 \)) and so (3.7) bounds \( \deg_L(C) \) absolutely. But by (3.5) we can choose \( L = L_i \) so that \( \deg_L(C) \) is arbitrarily large while (3.6) and hence (3.7) are still satisfied. This is a contradiction and so \( C \) is not an irreducible component of both \( Z_5(R_{1+2\delta}) \) and \( Z_6(R_{1+3\delta}) \). Let \( V \subset Z_{2\delta}(R_{1+3\delta}) \) be an irreducible component properly containing \( C \). Then \( \dim V \geq 2 \) and, in obtaining the upper bound for \( \kappa(L) \), we will get for
free that $V$ must be a surface. Applying Lemma 3.3 and arguing as in the proof of (3.3.1) gives

$$k \text{ times } V + \cdots + V \subset Z_{2k\delta}(R_{k(1+3\delta)}).$$

Since $A$ is simple $V + \cdots + V = A$ as soon as $k \geq \lceil g/\dim V \rceil$. Hence

$$(3.8) \quad \kappa(L) \leq (1 + 3\delta)\lceil g/\dim V \rceil \leq (1 + 3\delta)\lceil g/2 \rceil.$$ 

To see why $V$ must be a surface, note that by the Riemann-Roch theorem,

$$\kappa(L) \geq \sqrt{c_1(L)^2} \geq \sqrt{g^2}$$

so (3.8) would give a contradiction if $\dim V \geq 3$.

We obtain a lower bound for $\kappa(L)$ case by case according to the dimension. For $g = 2$, Theorem 1.2 is stronger than Theorem 1.3 so there is nothing to show. Suppose then that $g = 3$. To determine how many conditions are imposed on the linear series $[nL]$ in order to obtain a divisor of multiplicity $\alpha n$ for $\alpha \in \mathbb{R}^+$, it suffices to determine the possible tangent cones of degree $\leq \alpha n$ associated to divisors in $[nL]$. For $\alpha < \epsilon(L)$ all tangent cones are possible but once $\alpha > \epsilon(L)$ there are restrictions measured by $BS[nL \otimes I^n_{P(\alpha)}]$. In particular, $V \subset A$ is a divisor when $\dim A = 3$ and if $D \in [nL]$ with $\mult_P(D) = (1 + 3\delta)n + k$ then $D = kV + E$ for some effective divisor $E$ with

$$\mult_P(E) = (1 + 3\delta)n + k - k \mult_P(V) \leq (1 + 3\delta)n.$$ 

Hence $C_P(D) = kC_P(V) + C_P(E)$ and it is at most $h^0(\mathcal{O}_{P^3}((1 + 3\delta)n))$ conditions to kill all leading terms of degree $(1 + 3\delta)n + k$ for any $k \geq 0$. Adding everything up, it is at most

$$\frac{(1 + 3\delta)^3n^3}{6} + \frac{\alpha(1 + 3\delta)^2n^3}{2} + O(n^2)$$

conditions to impose multiplicity $1 + \alpha + 3\delta$ at $P$. Since $h^0(A, nL) \geq n^3 + O(n^2)$,

$$\kappa(L) \geq 1 + \alpha + 3\delta \quad \text{whenever} \quad \frac{(1 + 3\delta)^3}{6} + \frac{\alpha(1 + 3\delta)^2}{2} < 1.$$ 

A simple computation shows that $\kappa(L) \geq 9/4$ if $\delta \leq 1/30$ and this contradicts the upper bound $\kappa(L) \leq 2.2$ given by (3.8).
The case $g = 4$ is trivial since $\kappa(L) \geq \sqrt[4]{2.2}$, contradicting (3.8) as soon as $\delta \leq 1/30$. The case $g = 5$ can be handled just like the case $g = 3$. We know that $V + V \subset A$ is a divisor which appears in the base locus of $|nL \otimes T_P^{(\alpha)}|$ as soon as $\alpha \geq 2 + 6\delta$. Therefore, counting as in the case $g = 3$ gives that it is at most
\[
\frac{(2 + 6\delta)^5}{5!} n^5 + \frac{\alpha(2 + 6\delta)^4}{4!} n^5 + O(n^4)
\]
conditions to impose multiplicity $2 + \alpha + 6\delta$ at $P$. Riemann-Roch and (3.8) give a contradiction if, say, $\delta \leq 1/300$.

Last is the case where $g = 6$. In this case the lower bound $\sqrt[6]{6!}$ coming from Riemann-Roch falls just short of contradicting the upper bound (3.8) for $\delta \ll 1$. A slightly more efficient counting process gives the required bound. In particular, as above if $D \in |nL|$ and $\text{mult}_P(D) = (1 + \delta)n + k$ then $D$ vanishes to order at least $k$ along $C$. It follows that the tangent cone $C_P(D)$ has order at least $k$ along $C_P(C)$. Summing up the dimension of the spaces of tangent cones with these constraints gives that it is at most
\[
\frac{([3 + 9\delta] - (2 + 8\delta) - (1 + \delta)]n^6}{6!} + O(n^5)
\]
conditions to kill all leading terms of degrees between $(1 + \delta)n$ and $(3 + 9\delta)n$. It is at most an additional $(1 + \delta)^6n^6/6! + O(n^5)$ to kill leading terms of degree less than $(1 + \delta)n$. Once more, Riemann-Roch and (3.8) give a contradiction when $\delta \leq 1/300$ and this concludes the case $g = 6$.

To see why this proof of Theorem 1.3 is effective, choose $\delta$ sufficiently small ($\delta \leq 1/30$ for $g = 3, 4$ or $\delta \leq 1/300$ for $g = 5, 6$) so that the lower bound for $\kappa(L)$ is good enough to contradict (3.8). Then we claim that any curve $C$ satisfying (3.6) must be an irreducible component of both $Z_{\varepsilon}(R_{1+2\varepsilon})$ and $Z_{2\varepsilon}(R_{1+3\varepsilon})$. If not, then there is a surface $V \subset Z_{2\varepsilon}(R_{1+3\varepsilon})$; hence (3.8) holds and we obtain a contradiction. Thus (3.7) holds for any curve $C$ satisfying (3.6). The conclusion, using Theorem 2.1, is that there exists a constant $c(g)$ such that
\[
\frac{L.C}{\text{mult}_P(C)} \geq \frac{c(g) + 1}{c(g)} \quad \text{whenever} \quad \frac{L.C}{\text{mult}_P(C)} < 1 + \delta,
\]
giving an effective lower bound of $1 + \min\{\delta, 1/c(g)\}$ for $c(A)$.
REFERENCES

[CP] F. Campana and T. Peternell, Algebraicity of the ample cone of projective varieties, J. Reine Angew.
Math. 404 (1990), 160–166.
[D] J.-P. Demailly, Singular Hermitian metrics on positive line bundles, Complex Algebraic Varieties,
Lecture Notes in Math., vol. 1507 (Hulek, Peternell, Schneider, and Schreyer, eds.), Springer-
Verlag, New York, 1992, pp. 87–104.
[ELN] L. Ein, R. Lazarsfeld, and M. Nakamaye, Zero estimates, intersection theory, and a theorem of
Demailly, Proceedings of the Conference on Higher Dimensional Algebraic Geometry Trento
[Nak1] M. Nakamaye, Reducibility of principally polarized abelian varieties with highly singular polarization,
preprint.