Compactness

1. Suppose $S \subset X$ is an infinite subset of a compact topological space. Show that $S$ has a limit point in $X$.

2. Find all connected, compact subsets of $\mathbb{R}$ and show that your list is complete. Why would this be a much more complicated problem if $\mathbb{R}$ were replaced by $\mathbb{R}^2$?

3. Suppose $A$ and $B$ are closed, disjoint subsets of a metric space $X$. Is there necessarily an $\epsilon$-neighborhood of $A$ which is disjoint from $B$? What if one assumes that $A$ is compact?

4. Show that a continuous map $f : [0, 1] \to \mathbb{R}$ is uniformly continuous, that is, given any $\epsilon > 0$ there exists a $\delta > 0$ so that if $x, y \in [0, 1]$ and $|x - y| < \delta$ then $|f(x) - f(y)| < \epsilon$. Is this still true if $[0, 1]$ is replaced by $(0, 1)$? Which of the following functions, viewed as maps from $\mathbb{R}$ to $\mathbb{R}$, are uniformly continuous?
   a. $f(x) = |x|,$
   b. $g(x) = x^2,$
   c. $f(x) = \cos x,$
   d. $f(x) = e^x.$

5. Let $\mathbb{P}^1 = \mathbb{C} \cup \infty$ where $\infty$ is a symbol representing one element in $\mathbb{P}^1$. Define a topology on $\mathbb{P}^1$ as follows: a set $U \subset \mathbb{P}^1$ is open if and only if
   i. $U \cap \mathbb{C}$ is open in the regular metric topology on $\mathbb{C},$
   ii. If $\infty \in U$ then there exists some $N > 0$ so that $\{z \in \mathbb{C} : |z| > N\} \subset U.$
   
   The topological space $\mathbb{P}^1$ is called the projective line.
   a. Show that $\mathbb{P}^1$ is compact with this topology.
   b. How does $\mathbb{P}^1$ compare to the unit sphere in $\mathbb{R}^3$ with the regular topology inherited from $\mathbb{R}^3$?
   *c.* More abstractly, $\mathbb{P}^1$ can be viewed as the collection of lines in $\mathbb{C}^2$ which contain $(0, 0)$ and in fact it is the quotient of $\mathbb{C}^2 \setminus \{(0, 0)\}$ by the action of $\mathbb{C}^*$ given by $\lambda \cdot (a, b) = (\lambda a, \lambda b)$. Show that the topology on $\mathbb{P}^1$ is the quotient topology inherited from $\mathbb{C}^2 \setminus \{(0, 0)\}$ by this action.

6. Suppose $X$ is a metric space and $f : X \to X$ is a map from $X$ to itself. The map $f$ is called a contraction if there is a positive real number $\alpha < 1$ such that $d(f(x), f(y)) \leq \alpha d(x, y)$ for all pairs of points $x, y \in X$.
   a. For the following maps, from $\mathbb{R}$ to $\mathbb{R}$, identify which ones are contractions: $f(x) = x/2 + 1$, $g(x) = \cos x$, $h(x) = e^{-x}$.
b. Do these maps have any fixed points, that is points $x \in \mathbb{R}$ such that $f(x) = x$?

c. Suppose now that $X$ is compact. Show that any contraction has a unique fixed point. Give some example where $X = [0, 1]$ and find the fixed point for each of your examples.

7. This problem is about the famous Cantor set. The cantor set $C \subset [0, 1]$ is defined to be the set of all real numbers which can be expressed as

$$\sum_{n=1}^{\infty} \frac{a_n}{3^n}$$

where each $a_n$ is either 0 or 2.

a. Find those elements of the Cantor set which are given as finite sums

$$\sum_{n=1}^{k} \frac{a_n}{3^n},$$

where $k = 1, 2, 3, 4$. Can you see a pattern in how these sets of points are evolving?

b. Show that $C$ is totally disconnected.

c. Show that the complement of $C$ in $[0, 1]$ is a union of open intervals, whose lengths add up to 1.

d. Show that $C$ is compact.

e. Show that $C$ is an uncountable set.